Causality, Conjugate Points and Singularity Theorems in Space-time

by

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Abstract

Singularity theorems attempt to answer the question of when gravitation produces singularities. Scientists study what feasible and reasonable conditions would imply the existence of singularities in space-time. In the thesis, we will study two singularity theorems of Roger Penrose in [6] and of Hawking S.W. and Roger Penrose [8] respectively. We will show that strong energy condition, chronology condition, generic condition, existence of a trapped surface, a trapped set and a non-compact Cauchy surface are some physical conditions which will imply the existence of singularities.

摘要

物理物家們利用奇點理論研究時空因引力過大而扭曲的情況 ，他們主要關心時空扭曲的充分及合理的物理條件。 這篇論文將會討論R. Penrose 於1965年發表的奇點理論及 S.W. Hawking 和他於1970年發表的另一個奇點理論 ，這兩定理説明强能量條件、 時序條件、 一般性的條件、以及封閉陷獲面、 陷獲點集和 非緊的Cauchy 面的存在都是一些合理而又充分的物理條件。

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Contents

Chapter 1

Introduction

General Relativity predicts that our Universe contains 'holes' in which space and time are no longer meaningful. Physicists try to define the 'hole' in our space. 'Holes' represent singularities in space-time. Singularity theorems are to study the physical conditions for the existence of singularities in our space-time. In [10], it states that singularity theorems are interpreted as providing evidence of the classical singular beginning of the Universe and the singular final fate of some stars and the formation of black holes. Today, singularity theorems are still an active research topic in physics and mathematics.

In this thesis, it aims at proving two singularity theorems of R. Penrose in [6] and of S.W. Hawking and R. Penrose in [8] respectively.

Theorem[8] Space-time (M, g) cannot be null geodesic complete if

(1.) $R_{ab}K^aK^b \geq 0$ for all null vector K^a;

(2.) there is a non-compact Cauchy Surface K in M;

(3.) there is a trapped surface Γ in M.

Theorem[6] Space-time (M, g) is timelike or null geodesic incomplete if

(1.) there is no closed timelike curve [chronological condition];

(2.) $R_{ab}K^aK^b \geq 0$ for every causal vector K [strong energy condition];

(3.) any causal geodesic contains a point at which $\sum_{c,d=1}^{4} T^cT^dT_{[a}R_{b]cd[e}T_{f]} \neq 0$ where T is the tangent of the geodesic [generic condition];

(4.) there exists a compact achronal set without edge or a trapped surface.

For more details about the theorems, you can see Chapter 5.

Since singularity theorems are based on concepts of global causality, maximal causal curve, conjugate points and Raychaudhuri equation. The arrangement of the thesis are as follows:

First of all, we will briefly define some basic terms used in general relativity in Chapter 2.

Then, we will consider strongly causal and global hyperbolic space-time and study its properties, e.g. domain of dependence, in Chapter 3. The main part is to show that the existence of Cauchy surface implies M is globally hyperbolic.

Since there is a close relationship between the Lorentzian length of a future causal geodesic and the existence of singularity, in Chapter 4,we will study the topology in the space of causal curves. Then, we will define Jacobi fields and conjugate points. We will discuss that the relationship among Lorentzian length, chronology and conjugate points. Due to the importance of a conjugate point, we will study the conditions for its existence.

Finally, we will define singularities and prove two singularity theorems in Chapter 5. The two singularity theorems show that our universe should be b-incomplete. It means our Universe should contains ‘holes’.

Chapter 2

Basic Terminologies

Time-oriented Space-Time Manifold:

Let M be a smooth connected paracompact Hausdorff manifold of dimension 4 with a countable basis. Let g be a smooth symmetric tensor field of type $(0, 2)$ such that $g|_p$ is an inner product of signature $(1, 3)$[i.e., $(-, +, +, +)$]. If there is a smooth vector field X on M such that $g(X, X) < 0$, then (M, g) is said to be time oriented space-time manifold.

Timelike, Null, Spacelike, Causal or Future Causal Vectors:

A non-zero tangent vector $v \in T_P M$ is calassified as timelike, null, spacelike or causal if $g(v, v)$ is negative, zero, positive or non-positive respectively. A causal vector v is said to be future [past] if $g(v, X) < 0$ [> 0].

Convex Normal Neighbourhood:

An open set U is said to be a convex normal neighbourhood if for any p,q in U, there is the unique geodesic lying in U joining from p to q.

C^1 **Future Timelike ,Causal or Null-like Curve:**

A C^1 curve $\gamma : (a, b) \to M$ is said to be a future [past]directed non-spacelike , timelike or null-like curve if $\gamma'(t)$ is a future [past]non-spacelike, timelike or null-like vector for $t \in [a, b]$ respectively.

Continuous Future Timelike ,Causal or Null-like Curve:

A continuous curve $\gamma : (a,b) \rightarrow M$ is said to be a future [past]directed non-spacelike curve if for each $t_0 \in (a,b)$ there is an $\epsilon > 0$ and a convex normal neighbourhood $U(\gamma(t_0))$ of $\gamma(t_0)$ with $\gamma(t_0 - \epsilon, t_0 + \epsilon) \subseteq U(\gamma(t_0))$ such that given any t_1,t_2 with $(t_0 - \epsilon, t_0 + \epsilon)$, there is a smooth future directed non-spacelike curve in $(U(\gamma(t_0)), g_{U(\gamma(t_0))})$ from $\gamma(t_1)$ to $\gamma(t_2)$. Similarly, a continuous timelike and null-like curve can be defined.

We write $x \preceq y$ if there is a future continuous causal curve from x to y. Also, $x \ll y$ if there is a future continuous timelike curve from x to y.

Chronological Sets, I^+ and J^+:

$I^+(x)$ ={$y \in M$ |$x \ll y$} is called the chronological future of x; $I^-(x)$ ={$y \in M$ |$y \ll x$} is called the chronological past of x; $J^+(x)$ ={$y \in M$ |$x \preceq y$} is called the causal future of x; $I^+(x)$ ={$y \in M$ |$x \preceq y$} is called the causal past of x. The chronological or causal future of a set $S \subseteq M$ is defined by $I^+(S)$ ={$y \in M$ |$x \ll y$ for some $x \in S$}, $J^+(x)$ ={$y \in M$ |$x \preceq y$ for some $x \in S$}, respectively, and similarly for the pasts of $I^-(S)$ and $J^-(S)$.

Achronal Set:

A set $S \subseteq M$ is achronal if no two points of S are timelike related [i.e., if $x, y \in S$, then $x \not< < y$].

Edge of an Achronal Set:

Let S be achronal. edge(S) = {$p \in \overline{S}$ | every open neighbourhood O of p, $\exists q$ joint from p by a future timelike curve lying in O and r joint from p by a past timelike curve lying in O such that there exists a past time-like curve λ lying in O from q to r, but $\lambda \cap S = \emptyset$}.

Future Inextendible Casual Curve:

Let a continuous curve $\gamma : [a,b) \rightarrow M$ be a future non-space-like curve. If there exists $p \in M$ such that $lim_{t \rightarrow b^-} \gamma(t) = p$, then p is said to be a future end-point of γ. If γ has no future end-point, then γ is said to be future inextendible or future endless.

Limit Curve

$p \in M$ is said to be an accumulation point of a sequence of curve $\{\gamma_n\}$ if for any open neighbourhood O of p, there exists $N \in \mathbb{N}$ such that $\gamma_n \cap O \neq \emptyset$ for $n \geq N$. Also, $\gamma(t) : I \to M$ is said to be a limit curve of $\{\gamma_n\}$ if there exists a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ such that $\gamma(t)$ is an accumulation point of $\{\gamma_{n_k}\}$ for all $t \in I$ [i.e., for any $t \in I$, any open neighbourhood O of $\gamma(t)$, there exists $N \in \mathbb{N}$ such that $\gamma \cap O \neq \emptyset$ for all $n \geq N$].

Future-Distinguishing Space-Time:

(M, g) is future-distinguishing at $p \in M$ if $I^+(p) \neq I^+(q)$ for $q \neq p$ and $q \in M$.

Strong Causality Space-Time:

An open set $Q \subseteq M$ is causally convex if Q intersects no future causal curve in a disconnected set [if a future causal curve γ with $\gamma(0)$ and $\gamma(1) \in Q$, then $\gamma([0,1]) \in Q$]. Then, M is said to be strongly causal at p if p has arbitrarily small causally convex neighbourhoods [i.e. for any open O of p, there exists a causally convex neigbhourhood U_O of x such that $x \in U_O \subseteq O$]. M is said to be strongly causal if it is strongly causal at any point on M.

Domain of Dependence

Let S be an achronal subset of M. Define the future and past domains of dependence of S and the total domian of dependence of S, respectively, as follows:

$D^+(S) = \{x \in M \mid \text{every past endless causal curve from } x \text{ intersects } S\}$.

$D^-(S) = \{x \in M \mid \text{every future endless causal curve from } x \text{ intersects } S\}$.

$D(S) = \{x \in M \mid \text{every endless causal curve containing } x \text{ intersects } S\}$.

Clearly, $D(S) = D^+(S) \cup D^-(S)$.

$\widetilde{D^+}(S) = \{x \in M \mid \text{every past endless time-like curve from } x \text{ intersects } S\}$.

$\widetilde{D^-}(S) = \{x \in M \mid \text{every future endless time-like curve from } x \text{ intersects } S\}$.

$\widetilde{D}(S) = \{x \in M \mid \text{every endless time-like curve containing } x \text{ intersects } S\}$.

Clearly, $\widetilde{D}(S) = \widetilde{D^+}(S) \cup \widetilde{D^-}(S)$.

Cauchy Horizon

The future, past or total Cauchy horizon of an achronal closed set S is defined as (respectively):

$H^+(S) = \{x \in M|\ x \in \overline{D^+(S)}$ but $I^+(x) \cap D^+(S) = \emptyset\}$,

$H^-(S) = \{x \in M|\ x \in \overline{D^-(S)}$ but $I^-(x) \cap D^-(S) = \emptyset\}$,

$H(S) = H^+(S) \cup H^-(S)$

Cauchy Hypersurface:

A Cauchy hypersurface for M is an non-empty achronal set S for which $D(S) = M$.

Globally Hyperbolic Space-Time:

(M, g) is said to be globally hyperbolic if M is strongly causal and $J^+(u) \cap J^-(v)$ is compact for any u and $v \in M$.

Chapter 3

Causality in space-time

In section 3.1, we will state some basic facts in space-time. They are mainly about chronology and limit curve in space-time. Next, in section 3.2, we will discuss some global causality conditions. We mainly study two things. The first one is to show that a convex normal neighbourhood regarding as a manifold is causally convex. Also, we will discuss what are implications in geometry if strong causality fails at some points in M. Finally, in section 3.3, we will discuss globally hyperbolic space. We will show that the existence of Cauchy surface implies M is globally hyperbolic.

3.1 Preliminaries in space-time

On p.54-57 in [1], for any point in a space-time manifold, it admits an arbitrarily small convex normal neighbourhood containing it. Also, following p.103-105 in [7], if p and q can be joint by a future timelike <u>curve</u> lying in a convex normal neighbourhood, then they can be joint by a future timelike <u>geodesic</u> lying in it. The above statement is true if "timelike" is replaced by "causal". By this result, we can say for any future casual curve γ from p to q in a space-time manifold, for any neighbourhood O with $\gamma \subseteq O$, there exists a piecewise unions of future casual geodesic lying in O from p to q. Still, the statement is true if "geodesics" is replaced by "C^1 differentiable curves". It means, γ can be arbitrarily approximated by a piecewise differentiable curves. Also, referring to p.12-15 in [9], we have the following theorems.

Theorem 3.1.1. *Three basic properties in* (M, g)

(i.) $a \ll b$, $b \preceq b$ *implies* $a \ll b$*;*

(ii.) $a \preceq b$, $b \ll b$ *implies* $a \ll b$*;*

(iii.) Let $\widehat{ab}$ *and* $\widehat{bc}$ *be a future null geodesic from* a *to* b *and from* b *to* c *respectively. If the tangents of* $\widehat{ab}$ *and* $\widehat{bc}$ *are not collinear at* b*, then there is a future timelike curve from* a *to* c*.*

With the above theorem, we have some basic results about chronological sets.

(1.) $I^+(p)$ and $I^-(p)$ are open for any $a \in M$;

(2.) $I^+(S)$ and $I^-(S)$ are open for any $S \in M$;

(3.) $I^+(S) \subseteq J^+(S) \subseteq \overline{I^+(S)}$ and $I^-(S) \subseteq J^-(S) \subseteq \overline{I^-(S)}$ for any $S \subseteq M$;

(4.) $\partial J^+(S)$ is achronal.

We should notice that $J^+(S)$ may not be closed and $\overline{J^+(S)} = \overline{I^+(S)}$ for any $S \subseteq M$

Also, according to p.23 in [9], we have the following important lemma which is useful in singularity theorems.

Lemma 3.1.2. *For any* $S \subseteq M$*,* $\partial I^+(S)$ *is a topological* C^0 3*-manifold without boundary.*

Next, we will have a limit curve theorem to study the convergence of a family of causal curves. It is proved in the appendix.

Theorem 3.1.3. [*limit curve theorem*] *Let $\{\gamma_n\}$ be a sequence of future inextendible causal curves in (M, g). If p is an accumulation point of the sequence $\{\gamma_n\}$, then there is a future causal curve γ which is a limit curve of the sequence γ_n such that $p \in \gamma$ and γ is future inextendible.*

According to p.372 in [5] and p.194 in [13], we can say something about $\partial J^+(S)$ for any $S \subseteq M$.

Corollary 3.1.4. *Let S be a subset of M. For any $p \in \partial J^+(S) - \overline{S}$, there exists a past null geodesic segment lying on $\partial J^+(S)$ such that it is either past endless on $\partial J^+(S)$ or has a past end-point on edge(S).*

3.2 Global causality condition

First, let Q be an open subset of M and let $x, y \in Q$. Then we write $x \ll_q y$ if and only if a future timelike curve lying in Q exists from x to y, and $x \preceq_Q y$ if and only if a future causal curve in Q exists from x to y. If the open Q is connected, it is a space-time manifold in its own right. Hence, all the properties in section 3.1 hold. Let we define $\langle x, y\rangle_Q = \{z \in M | x \ll_Q z \ll_Q y\}$. If Q is open, then the sets $\langle x, y\rangle_Q$ is open where $x, y \in Q$.

Proposition 3.2.1. *If N is a convex normal neigbhourhood and $x, y \in N$, then the set $\langle x, y\rangle_N$ has the property that no future causal curve lying in N can intersect $\langle x, y\rangle_N$ in a disconnected set.*

Proof. For $u, v \in \langle x, y\rangle_N$ and $u \preceq_N v$, we let $\eta : [0, 1] \to N$ be a future causal curve from $u = \eta(0)$ to $v = \eta(1)$ lying in N. For each $s \in [0, 1]$, we have $x \ll_N u \preceq_N \eta(s) \preceq_N v \ll_N y$ and $\eta(s) \in N$. The first paragraph of the section

3.1 states that there are two future timelike geodesics lying in N joining from x to $\eta(s)$ and from $\eta(s)$ to y respectively. Hence, $\eta(s) \in \langle x, y\rangle_N$. $\square$

Proposition 3.2.2. *If N is a convex normal neigbhourhood and Q is an open set contained in N and $p \in Q$, then there exists u, v such that $p \in \langle u, v\rangle_N \subseteq Q$.*

Proof. First, we choose a coordinate neigbhourhood (t, x_1, x_2, x_3) of p in N such that $g|_p = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$ and $\frac{\partial}{\partial t}$ is pointing future at p.
For any open $Q \subseteq N$, there exists a small open ball, B, on $\mathbb{R}^4$, such that

(I.) $p \in \widetilde{B} = exp_p(B) \subseteq Q$;

(II.) for any future timelike curve, $\beta(s) = exp_p(t(s), x_1(s), x_2(s), x_3(s)) \subseteq \widetilde{B}$ with respect to g and timelike vector field T, $\beta(s)$ is also a future timelike curve under a metric $\widetilde{g} = -4dt^2 + dx_1^2 + dx_2^2 + dx_3^2$ and a timelike vector field $\frac{\partial}{\partial t}$.

Let W be a convex normal neighbourhood of $p \subset\subset \widetilde{B}$ under g. Since W is open, there exists a $\delta > 0$, such that $p \in E = \{exp_p(t, x_1, x_2, x_3) \mid |t| < \frac{\delta}{2}, x_1^2 + x_2^2 + x_3^2 < 4\delta^2\} \subseteq W$. We take $u = exp_p(-\frac{\delta}{2}, 0, 0, 0)$, $v = exp_p(\frac{\delta}{2}, 0, 0, 0)$. Also, $(W, \widetilde{g}, \frac{\partial}{\partial t})$ is a flat space, so under $\widetilde{g}$, $\frac{\partial}{\partial t}$, $\langle u, v\rangle_W$=$\{exp_p(t, x_1, x_2, x_3)| -4(t-\frac{\delta}{2})^2 + x_1^2 + x_2^2 + x_3^2 < 0\}$ $\cap$ $\{exp_p(t, x_1, x_2, x_3)|t > -\frac{\delta}{2}\}$ $\cap$ $\{exp_p(t, x_1, x_2, x_3)| -4(t+\frac{\delta}{2})^2 + x_1^2 + x_2^2 + x_3^2 < 0\}$ $\cap$ $\{exp_p(t, x_1, x_2, x_3)|t < \frac{\delta}{2}\}$. We denote the set as $\langle u, v\rangle_{(W, g, \frac{\partial}{\partial t})}$. It is easy to show that $\langle u, v\rangle_{(W, g, \frac{\partial}{\partial t})}$ is inside W.
Finally,we will claim that under g and T, $p \in \langle u, v\rangle_N \subseteq W(\subseteq \widetilde{B} \subseteq Q)$ under g and T. If the claim is true, the prosposition is done.
Proof of the Claim: Suppose it is false, there exists $\eta \in N$ such that $\eta \in \langle u, v\rangle_N$ but not in W. Then for any future timelike curve $\beta(s) : [0, 1] \to N$, with $\beta(0) = u$, $\beta(\frac{1}{2}) = \eta$, $\beta(1) = v$, we have $\beta(\frac{1}{2}) \notin W$. We let s_0 =inf$\{s \in [0, 1]|\beta(s) \in \partial W\} \in (0, 1)$. As $\beta(s_0) \in \partial W$ and $\beta([0, s_0)) \in W$, $\beta([0, s_0]$ must cut $\partial\langle u, v\rangle_{(W, \widetilde{g}, \frac{\partial}{\partial t})}$ for some $\xi \in (0, s_0)$. By (II), $\beta(s) \in I^+(u)$ under $\widetilde{g}$ and $\frac{\partial}{\partial t}$. It means that there is $\beta(\xi) \in \partial I^-(v)$ under $\widetilde{g}$ and $\frac{\partial}{\partial t}$. Hence, there is no past timelike curve lying in W

from v to $\beta(\xi)$ under $\widetilde{g}$ and $\frac{\partial}{\partial t}$. As a result, by (II), $\beta(\xi) \not<\!\!<_W v$ under g and T. By the Proposition 3.2.1, it means the geodesic lying in W from $\beta(\xi)$ to v must not be a future timelike curve. Since the geodesic lying in N from $\beta(\xi)$ to v is the same as that lying in W, we have $\beta(\xi) \not<\!\!<_N v$ under g and T. Contradiction. $\square$

Finally, there is a summary. By the Propositions 3.2.1 and 3.2.2, we have an important result below

Theorem 3.2.3. *Any convex normal neigbhourhood, if regarded as a space-time manifold in its own right, must be strongly causal.*

Next, we will show one useful lemma to determine the necessary and sufficient condition for strong causality fails at a point.

Definition 3.2.4. *A local causality neighbourhood is a causality convex open set whose closure is contained in a convex normal neighbourhood in M.*

Proposition 3.2.5. *M is strongly causal at p if and only if p is contained in some local causality neighbourhood.*

Proof. ($\Rightarrow$) Let N be a convex normal neighbourhood of p. As M is strongly causal at p, there exists a convex normal neighbourhood $Q \subset\subset N$ such that $\overline{Q} \subseteq N$. By definition, Q is a local causality neighbourhood of p.
($\Leftarrow$) Let Q be a local causality neighbourhood at p. There exists a convex normal neighbourhood N of p such that $\overline{Q} \subseteq N$. For any small open neighbourhood E of p with $E \subseteq Q$, by the Proposition 3.2.2, there exists u,$v \in E$ such that $p \in \langle u, v\rangle_N \subseteq E \subseteq Q \subset\subset N$. Suppose there exists a future causal curve β cutting $\langle u, v\rangle_N$ in a disconnected set. By the Proposition 3.2.1, $\beta \nsubseteq N$. However, Q is causally convex means $\beta \subseteq Q$. Hence, $Q \nsubseteq N$. Q is not a local causality neighbourhood. Contradiction. $\square$

Lemma 3.2.6. *Let $p \in M$. Then strong causality fails at p if and only if there exists $q \preceq p$, with $q \neq p$, such that $x \ll p$ and $q \ll y$ together imply $x \ll y$ for all such $x, y \in M$.*

Proof. ($\Rightarrow$) Suppose strong causality fails at p. Let N be a convex normal neighbourhood of p. According to the proof in the Proposition 3.2.2, there exists $p \in Q_i = \langle u_i, v_i \rangle_N \subseteq N$ with $Q_{i+1} \subseteq Q_i$, $\overline{Q_i} \subset N$ and $\cap_{i=1}^{\infty} Q_i = p$. Indeed, by the Proposition 3.2.5, Q_i is not a local causality neighbourhood. For each i, there exists a future causal curve $\gamma_i : [0, 1] \to M$ which intersects Q_i in a disconnected set. By the Proposition 3.2.1, $\gamma_i \nsubseteq N$. We let $\gamma_i(0) = a_i \in Q_i$, $b_i = \gamma_i(s_i)$ is the first point on ∂N, $c_i = \gamma_i(t_i)$ is the last point on ∂N, $d_i = \gamma_i(1) \in Q_i$. As ∂N is compact, we assume $\{c_i\}$ converges to $q \in \partial N$. Also, $\cap Q_i = \{p\}$ implies both $\{a_i\}$ and $\{d_i\}$ converges to p.

The geodesic from c_i to d_i lying in N is future causal. Hence, the geodesic from c to p lying in N is also future and causal [by smoothness of $\exp_u(v)$ with respect to u and v]. As a result, we can let $\gamma : [0, 1] \to N$ be a future causal curve such that $\gamma(0) = c$, $\gamma(\frac{1}{2}) = q$ and $\gamma(1) = p$. If $x \ll p$ and $q \ll y$, then $p \in I^+(x)$. $I^+(x)$ is open and $a_k \to p$, there exists k_1 such that $a_i \in I^+(x)$ for $i \geq k_1$. Also $c \preceq q \ll y$. Hence, $c \in I^-(y)$. Similarly, there exists $k_2 \in N$ such that $c_i \in I^-(y)$ for $i \geq k_2$. Take $k = max\{k_1, k_2\}$, $x \ll a_k \ll b_k \ll c_k \ll y$. Thus, $x \ll y$.

($\Leftarrow$) Let $p \in P$, $q \in Q$ be two disjoint open sets. It suffices to show that P cannot be causally convex as P can be arbitrarily small. Take $x \in P \cap I^-(p)$ and $z \in P \cap I^+(p)$. We have $q \preceq p \ll z$. It means $q \in I^-(z)$. $q \in Q \cap I^-(z)$ is an open set containing q. There exists y in $Q \cap I^-(z) \cap I^-(q)$. Since $x \ll p$ and $q \ll y$, we have $x \ll y \ll z$. As a result, there exists a future causal curve from x to z passing through y. where $x, z \in P$ but $y \in Q$. As $y \notin P$, P is not causally convex. $\square$

Finally, we will show a useful lemma about causality failure.

Definition 3.2.7. *A point $p \in M$, through which passes a closed timelike curve, is called vicious.[i.e.$p \in I^+(p)$]. Denote the set of all vicious points of M by V. It is clear that $V = \cup_{x \in M}(I^+(x) \cap I^-(x))$ and it is open.*

Lemma 3.2.8. *If future-distinction fails at $p \notin V$, then p lies on a past endless null geodesic $\gamma \subseteq V^c$ along which future distinction fails. Moreover, we will show that $I^+(p) = I^+(r)$ for each $r \in \gamma$.*

Proof. : It suffices to show that there exists a past-endless null geodesic from p on $\partial I^+(p)$.

In the first place, we construct a segment of past-endless null geodesic from p. Since $p \notin V$, it means $p \notin I^+(p)$. So, $p \in \partial I^+(p)$. As future-distinction fails at p, there exists $q \neq p$ such that $I^+(p) = I^+(q)$. Then, let d be a distance function on M respect to a Riemannian metric, there exists a $\epsilon > 0$ such that $q \notin B_\epsilon(p)$ with respect to d. Let $\{p_n\}$ be a sequence such that $p_n \in B_\epsilon(p) \cap I^+(p)$ and $p_n \to p$. Let γ_n which is arc-length parametrization with respect to d be a past piecewise differentiable timelike curve from $\gamma_n(0) = p_n$ to q. We extend γ to be past endless. By the Theorem 3.1.3, there exists a past endless causal curve γ from p. We claim γ is a past null geodesic from p on $\partial I^+(p)$. $\gamma_n([0, \frac{\epsilon}{2}]) \subseteq B_\epsilon(p) \cap I^+(q)$ for n large, Hence,$\gamma([0, \frac{\epsilon}{2}]) \subseteq \overline{I^+(q)} = \overline{I^+(p)}$. Suppose there exists $t_0 \in (0, \frac{\epsilon}{2})$ such that $\gamma(t_0) \in I^+(p)$. We have $p = \gamma(0) \succeq \gamma(t_0) \gg p$ which says $p \in V$. Contradiction. So, $\gamma([0, \frac{\epsilon}{2}]) \subseteq \partial I^+(q) = \partial I^+(p)$. $\gamma([0, \frac{\epsilon}{2}])$ is a null geodesic. Otherwise, when $\gamma(0)$ and $\gamma(\frac{\epsilon}{2})$ is joint by piecewise past causal geodesics, there are two cases: (1.) there is one geodesic is timelike, then by the Theorem 3.1.1, $\gamma(0) \gg \gamma(\frac{\epsilon}{2})$ which contradicts $\partial I^+(p)$ is achronal. (2.) If all are null geodesics, then their union must not be a single geodesic. Hence, by the Theorem 3.1.1 (iii.), $\gamma(0) \in I^+(p)$. We still have contradiction. The claim is done.

Next, we extend γ to be the past endless null geodesic from p and show $\gamma \subseteq \partial I^+(p)$. Suppose there exists $s > 0$ such that $\gamma([0, s]) \subseteq \partial I^+(p)$, but there exists

$t_n \to s^+$ such that $\gamma(t_n) \notin I^+(p)$. We claim $\gamma(s)$ is q. If $\gamma(s) \neq q$, then we have $I^+(\gamma(s)) \supseteq I^+(p)$ as $\gamma(s) \preceq \gamma(0) = p$. On the other hand, $\gamma(s) \in \partial I^+(p)$. Hence, $I^+(\gamma(s)) \subseteq I^+(p)$. As a result, we have $I^+(\gamma(s)) = I^+(q)$. Thus, by the above construction, there exists a null geodesic $\eta[s, s_1] \subseteq \partial I^+(p)$ from $\gamma(s)$ for some $s_1 > 0$. By the assumption about s, $\gamma \cup \eta$ is not a single null geodesic. However, it lead to $\eta(s_1) \ll \gamma(0)$ which contradicts the $\partial I^+(p)$ is achronal. The claim is done. Since, $I^+(p) = I^+(q)$, by the above construction again, we can construct a past null geodesic $\alpha \subseteq \partial I^+(p)$ from q. By similar argument, there is a contradiction with the achronal property of $\partial I^+(p)$. Hence, the whole γ is on $\partial(I^+(p))$.

Finally, we show that $\gamma(t) \in V^c$. By the above method, it is easy to show $I^+(p) = I^+(\gamma(t))$ and $\gamma(t) \in \partial I^+(p)$. Hence, $\gamma(t) \notin I^+(\gamma(t))$ and $\gamma(t) \notin V$.

□

Finally, we can prove the following theorem.

Theorem 3.2.9. *Suppose strong causality fails at p. Then at least one of the following holds:*

(a) *there are closed timelike curves through p. [i.e. $p \in V$];*

(b) *p lies on a past-endless null geodesic on ∂V, at every point of which future-distinction fails;*

(c) *p lies on a future-endless null geodesic on ∂V, at every point of which past-distinction fails;*

(d) *p lies on both a past-endless null geodesic on ∂V along which future-distinction fails and a future-endless null geodesic on ∂V along which past-distinction fails, except that at p itself may not be both past-and future-distinguishing;*

(e) an endless null geodesic γ through p exists, at every point of which strong causality fails, such that if u and v are any two points of γ with $u \preceq v, u \neq v$, then $u \ll x$ and $y \ll v$ together will imply $y \ll x$.

Proof. Let N be a convex normal neighbourhood containing p. As strong causality fails at p. It is valid to let $a_i, b_i, c_i, d_i, \gamma_i$ and Q_i which are all the same in the Proposition 3.2.6 here. Clearly, $a_i \preceq_N b_i \preceq c_i \preceq_N d_i$. Hence, there are two future causal geodesics lying in N from a_i to b_i and c_i to d_i. Let say they are $\widehat{a_i b_i}$ and $\widehat{c_i d_i}$. As both a_i and d_i converge to p and W.L.O.G., b_i and c_i can be assumed to converge to b and c respectively, $\widehat{pb}$, $\widehat{cp}$ are future causal geodesic.

Case (1): Both $\widehat{pb}$ and $\widehat{cp}$ are future timelike geodesic.

Since $b \in I^+(p)$ and $c \in I^-(p)$. Hence, $b_i \in I^+(p)$ and $c_i \in I^-(p)$ for large i. As a result, $p \ll b_i \preceq c_i \ll p$. The condition (a) holds.

Case (2): $\widehat{pb}$ is a future timelike geodesic and $\widehat{cp}$ is a future null-like geodesic.

We let $x_n \in I^+(p) \cap I^-(b)$ such that x_n converges to p. We have two relations: (i.) $p \ll x_n \ll b$ and (ii.) $c \preceq p$. Hence, $c \ll x_n$. If n is fixed, $c_i \ll x_n \ll b_i$ for large i. Also, according to the construction of b_i and c_i, we have (iii.) $x_n \ll b_i \preceq c_i \ll x_n$. Hence, $x_n \in V$. Hence, $p \in \overline{V}$. For $p \in V$, then the condition (a.) holds. For $p \notin V$, it means $p \in \partial V$. We claim future distinction fails at p. For $y \in I^+(c)$, by (i.) and (iii.), we have $p \ll x_n \ll c \ll y$. Hence, $y \in I^+(p)$ and $I^+(c) \subseteq I^+(p)$. On the other hand, by (ii.), we have $I^+(c) \supseteq I^+(p)$. The claim is done. By the Lemma 3.2.8, there exists a past endless null geodesic $\gamma \subseteq V^c$ from p along which future distinction fails. It remains to show $\gamma \subseteq \partial V$. Since $I^+(\gamma(t)) = I^+(p)$. Hence, for each t, there exists a sequence of $z_k \in I^+(p) \cap I^-(b)$ such that $z_k \to \gamma(t)$. Then, we have $c \ll z_k \ll b$. Hence, for i large, $z_k \ll b_i \preceq c_i \ll z_k$. It means $z_k \in V$. So, $\gamma(t) \in \partial V$. We have the condition (b).

Case (3): $\widehat{pb}$ is a future null geodesic while $\widehat{cp}$ is a future timelike geodesic.

Suppose T is the original time vector field in M. We take $\widetilde{T} = -T$ to be a new time vector field in M. So, under $\widetilde{T}$, $\widehat{pb}$ becomes a past null geodesic and $\widehat{cp}$ become a past timelike geodesic. By the case ii, p lies on a past-endless null geodesic on ∂V, at every point of which future-distinction fails under $\widetilde{T}$. So, if we consider T again, the condition (c) holds.

Case (4): $\widehat{pb}$ and $\widehat{cp}$ are a future null geodesic but their directions are different at p.

We have $c \ll b$ and $\widehat{pb}$ and $\widehat{cp} \subseteq \overline{I^+(c)} \cap \overline{I^-(b)} \subseteq \overline{V}$. We need to consider the following four situations.

If $p \in V$, then the condition (a) is resulted.

If there exists some $r \in \widehat{pb}$ such that $r \in V$ and $r \neq p$, then $p \preceq r \ll r \preceq b$. So, $p \ll b$. Following the proof of the case (2), we have the conditions (b) holds.

If there exists some $r \in \widehat{cp}$ such that $r \in V$ and $r \neq q$, then it is similar to the case (3). It is easy to prove that the condition (c) holds.

If both $\widehat{pb}$ and $\widehat{cp} \subseteq (\partial V - V)$, then for any $r_1 \in \widehat{cp} - \{p\}$, we can take $\widehat{r_1 b}$ to be a future timelike geodesic from r_1 to b and $\widehat{cr_1}$ to be a future null geodesic from c to r_1. We replace the role of p in the proof of the case (2) with r_1. We have r_1 lies on a past endless null geodesic, called η_{r_1}, on ∂V at every point of which future distinction fails. Moreover, *eta* must include $\widehat{cr_1}$. Otherwise, we have $r_1 \in V$ which has a contradiction. Similarly, for any $r_2 \in \widehat{pb} - \{p\}$, we can take $\widehat{r_2 b}$ to be a future null geodesic from r_1 to b and $\widehat{cr_2}$ to be a future timelike geodesic from c to r_1. Following the proof of the case (3.), we have r_2 lies on a future endless null geodesic on ∂V at every point of which past distinction fails. Also the null geodesic contains $\widehat{r_2 b}$. Combining the two results, the condition (d) is resulted.

Case (5): both $\widehat{pb}$ and $\widehat{cp}$ are future null geodesic which their directions at p are the same.

First, we claim strong causality fails along $\widehat{pb}$ such that if u and v are any two points of $\widehat{pb}$ with $u \preceq v, u \neq v$, then $u \ll x$ and $y \ll v$ together will imply $y \ll x$. We have $u \preceq v$. If $y \ll v$ and $u \ll x$, we have that $y \ll v \preceq b$ implies $y \ll b_i$ and $p \preceq u \ll x$ implies $d_i \ll x$. So $y \ll x$. By the Lemma 3.2.6, strong causality fails at u and v. The claim is done. Similarly, strong causality fails along $\widehat{cp}$. As strong causality fails at b. Hence, for a convex normal neighbourhood $\widetilde{N}$ of b. Let $\widetilde{Q_i} = \langle \widetilde{v_i}, \widetilde{u_i} \rangle_{\widetilde{N}}$ such that $\cap_{i=1}^{\infty} \widetilde{Q_i} = b$ and $\widetilde{Q}_{i+1} \subseteq \widetilde{Q_i}$. We have a future causal curve $\widetilde{\gamma_i} : [0,1] \to M$ meeting $\widetilde{Q_i}$ in a disconnected set. $\widetilde{\gamma_i} \nsubseteq \widetilde{N}$. We let $\widetilde{\gamma_i}(0) = \widetilde{a_i} \in \widetilde{Q_i}$, $\widetilde{b_i} = \widetilde{\gamma_i}(s_i)$ is the first point on $\partial \widetilde{N}$, $\widetilde{c_i} = \widetilde{\gamma_i}(t_i)$ is the last point on $\partial \widetilde{N}$, $\widetilde{d_i} = \widetilde{\gamma_i}(1) \in \widetilde{Q_i}$. So, $\widetilde{a_i}$, $\widetilde{d_i} \to b$, $\widetilde{b_i} \to B_1$ and $\widehat{bB_1}$ is future causal geodesic. We need to consider the following two situations:

If the directions of $\widehat{pb}$ and $\widehat{bB_1}$ are different at b, then $p \ll \widetilde{B_1}$ and there exists a sequence $\{x_n\} \in I^+(p) \cap I^-(B_1)$ such that $x_n \to p$ and $x_n \notin \partial I^-(b)$. So $p \ll x_n \ll B_1$ implies $x_n \ll \widetilde{b_i} \ll \widetilde{d_i}$. Hence, $x_n \ll b$ and $p \ll x_n \ll b_i \ll c_i$ for large i. So for any $y \in I^+(c)$, we have $c_i \ll y$ for large i. Thus $I^+(p) \supseteq I^+(c)$. Also, $c \preceq p$ means $I^+(p) \subseteq I^+(c)$. We have $I^+(p) = I^+(c)$. By the Lemma 3.2.8, the conditions (b) is resulted.

If $\widehat{pb} \cup \widehat{bB_1}$ is a single null geodesic, by the method in the first paragraph in the case (5.), it is easy to show strong causality fails along it such that if u and v are any two points of $\widehat{pb} \cup \widehat{bB_1}$ with $u \preceq v, u \neq v$, then $u \ll x$ and $y \ll v$ together will imply $y \ll x$. Then, we repeat the above process. If this process does not terminate, we have a future endless null geodesic along which strong causality fails. Indeed, if $\widehat{cp}$ has the same situation, as $\widehat{cp} \cup \widehat{pb}$ is a single null geodesic. As a result, there exists an endless null geodesic along which strong causality fails. $\square$

3.3 Domains of Dependence

In PDE, a point p is in the domain of dependence of S if the state of any system at p can be completely specified by initial condition on S. As a signal must travel along time-like or null-like curve, we should expect that the initial data on S would completely determine the situation at p if and only if every such curve from p strikes S. Thus, we have a natural definition of domain of dependence in general relativity which is mentioned on p.9 in Chapter 2.

First, we discuss $\widetilde{D^+}(S)$. It is clear that $S \subseteq D^+(S)$.

Proposition 3.3.1. *If S is achronal, then $\overline{\widetilde{D^+}}(S)$=$\widetilde{D^+}(S) \cup \overline{S}$.*

Proof. It is easy to show that $\overline{\widetilde{D^+}}(S) \supseteq \widetilde{D^+}(S) \cup \overline{S}$. So it suffices to prove the converse. For any $p \in \overline{\widetilde{D^+}}(S) - \overline{S}$, there exists $p_n \in \widetilde{D^+}(S)$ such that $p_n \to p$. There exists a convex normal neighbourhood O of p such that $O \cap \overline{S} = \emptyset$. For any past endless timelike curve γ from p, there exists $\gamma(t_0) \in O$ such that $\gamma(t_0) \in I^-(p)$. Then, $\gamma(t_0) \in I^-(p_n)$ for large n. Thus, $\widehat{p_n\gamma(t_0)}$ is a past timelike geodesic in O with $\widehat{p_n\gamma(t_0)} \cap S = \emptyset$. As $p_n \in \widetilde{D^+}(S)$ and $p_n\gamma(t_0) \cup \gamma|_{[t_0,\infty]}$ is a past endless timelike curve from p_n. We have $\gamma|_{[t_0,\infty]} \cap S \neq \emptyset$. It means $\gamma|_{[0,\infty)} \cap S \neq \emptyset$ and $\gamma(0) = p$. Hence, $p \in \widetilde{D^+}(S)$. $\square$

Corollary 3.3.2. *If S is achronal and closed, $\widetilde{D^+}(S)$ is closed.*

Next, we will show the relationship between $\widetilde{D^+}(S)$ and $D^+(S)$. It is clear that $\widetilde{D^+}(S) \subseteq D^+(S)$

Lemma 3.3.3. *Let d be a distance with respect to a Riemannian metric. For $q \in M$ and a convex normal neighbourhood U of q, if $p \neq q \in I^+(q) \cap U$ with $d(p,q) < 1$, then for a past causal curve λ from q, we have a past time-like curve $\gamma(t)$ with $\lambda(t) \in I^-(\gamma(t))$ and $d(\gamma(t), \lambda(t)) < \frac{4}{1+t}$*

Proof. Suppose $\lambda : [0, \infty) \to M$ with $q = \lambda(0)$. We first consider $\lambda|_{[0,1]}$ only. Let U_t be a convex normal neighbourhood of $\lambda(t)$ with $U_t \subseteq B_{\frac{1}{1+t}}(\lambda(t))$. $\lambda|_{[0,1]}$ is covered with a finite number of U_t. Let say they are $U_{t_1}, U_{t_2}, ..., U_{t_n}$ with $0 = t_1 < t_2 < ... < t_n \leq 1$.

Since $q = \lambda(0) \in U_{t_1}$. We can let $\lambda(s_1)$ be the first point on ∂U_{t_1}. Then $\lambda(s_1) \in U_{t_2}$. For any $p \neq q \in I^+(q) \cap U_{t_1}$, there exists a past timelike curve $\gamma_1 : [0, 2s_1] \to U_{t_1}$ with $p = \gamma_1(0)$, $\lambda(s_1) = \gamma_1(2s_1)$ and $\gamma_1(s_1) \in U_{t_1} \cap U_{t_2}$. We can let $\lambda(s_2)$ be the first point on ∂U_{t_2}. Then $\lambda(s_2) \in U_{t_3}$. There exists a past timelike cruve $\gamma_2 : [s_1, 2s_2] \to U_{t_2}$ with $\gamma_1(s_1) = \gamma_2(s_1)$, $\lambda(s_2) = \gamma_2(2s_2)$ and $\gamma_2(s_2) \in U_{t_2} \cap U_{t_3}$. W.L.O.G., we can assume $\lambda(1) \in U_{t_n}$. Then we repeat the above process in n times until we have $\gamma_n : [s_{n-1}, 2s_n] \to U_{t_n}$ with $\gamma_{n-1}(s_{n-1}) = \gamma_n(s_{n-1})$, $\lambda(1) = \gamma_n(2s_n)$ and $\gamma_n(s_n) \in U_{t_{n-1}} \cap U_{t_n}$. [Note: $s_n = 1$]

Let $s_0 = 0$ and $\gamma : [0, 1] \to$ with $\gamma(t) = \gamma_m(t)$ where $s_{m-1} \leq t \leq s_m$. Then, γ is a timelike curve with $\lambda(t) \in I^-(\gamma(t))$. Also, the large radius of the cover balls is 1. When we consider $\frac{c}{1+t} > 1$ for $t \in [0, 1]$, we can set $c = 2$. Hence, for any $t \in [0, 1]$, there exists m such that $s_{m-1} \leq t \leq s_m$, we have $d(\gamma(t), \lambda(t)) < d(\gamma(t), \lambda(s_m)) + d(\lambda(s_m), \lambda(t)) < \frac{4}{1+t}$ for $t \in [0, 1]$.

We repeat the above process for $\lambda_{[n,n+1]}$ where n is an integer. We can construct the required past timelike curve γ. □

Proposition 3.3.4. *If S is achronal and closed, $\widetilde{D^+}(S) = \overline{D^+}(S)$*

Proof. It is easy to show that $\widetilde{D^+}(S) \supseteq \overline{D^+}(S)$ by the Corollary 3.3.2. It suffices to show the converse.

For any $p \in \widetilde{D^+}(S)$, there are two cases to consider.

Case (i.) $p \in S$. So, $p \in D^+(S)$.

Case (ii.) $p \notin S$. It implies $p \in I^+(S)$. For $q \in I^-(p) \cap I^+(S)$ with $d(p, q) < 1$ where d is a distance on M with respect to a Riemannain metric. We claim $q \in D^+(S)$.

First, we show $q \in \text{int}(\widetilde{D^+}(S))$. For any $r \in I^+(S) \cap I^-(p)$, $r \in I^+(S)$ means there exists $s_0 \in S$ and a past timelike curve η from r to s_0. Since $r \in I^-(p)$, there exists a past timelike curve $\widehat{pr}$ from p to r and $\widehat{pr} \cup \eta$ is a past timelike curve from p which meets s_0. S is achronal means $\widehat{pr} \cap S = \emptyset$. Then, for any past endless causal curve γ from r, $\widehat{pr} \cup \gamma$ is a past endless causal curve from p. So, $\gamma \cap S \neq \emptyset$. It means $r \in \widetilde{D^+}(S)$. As $q \in I^+(S) \cap I^-(p) \subseteq \widetilde{D^+}(S)$, we have $q \in \mathrm{int}(\widetilde{D^+}(S))$.

Then, for any past endless causal curve λ from $q = \lambda(0)$, we will As $d(p,q) < 1$, by the Lemma 3.3.3, there exists a past timelike curve $\gamma(t)$ from p such that (a.) $\lambda(t) \in I^-(\gamma(t))$ and (b.) $d(\gamma(t), \lambda(t)) < \frac{4}{1+t}$. Since $\lambda(t)$ is past-endless, by $(b.)$, $\gamma(t)$ is past endless, too. Since $p \in \widetilde{D^+}(S)$ and $\gamma(t)$ is a past endless timelike curve from p. There exists $t_0 > 0$ such that $\lambda(t_0) \in S$. By $(a.)$, we have $\lambda(t_0) \in I^-(S)$. Since S is achronal, $\lambda(t_0) \notin \widetilde{D^+}(S)$. Also, $\lambda(0) = q \in \mathrm{int}(\widetilde{D^+}(S))$. There exists $s_0 < t_0$ such that $\lambda(s_0)$ is the first point lying on $\partial\widetilde{D^+}(S)$.

Next, we will show $\lambda(s_0) \notin I^+(S)$. Suppose $\lambda(s_0) \in I^+(S)$. Since $\lambda(s_0) \preceq q \ll p$, $\lambda(s_0) \in I^-(p) \cap I^+(S)$. Following the proof of $q \in \mathrm{int}(\widetilde{D^+}(S))$, it is easy to show that $\lambda(s_0) \in \mathrm{int}(\widetilde{D^+}(S))$. Contradiction.

Finally, we will show $q \in D^+(S)$. Since $\lambda(s_0) \in \partial\widetilde{D^+}(S)$ and by the Corollary 3.3.2, we have $\lambda(s_0) \in \widetilde{D^+}(S)$. Suppose $\lambda(s_0) \notin S$. S is achronal, so we have $\lambda(s_0) \in I^+(S)$ which has contradiction. So, $\lambda(s_0) \in S$ and $q \in D^+(S)$.

As a result, we can construct a sequence $\{q_n\} \in D^+(S)$ which converges to p. $\square$

Referring to p.41 and p.42 in [9], we have the following proposition.

Proposition 3.3.5. *If S is achronal, then*

(1.) $S \subseteq D^+(S)$*;*

(2.) $H^+(S)$*is closed and achronal;*

(3.) If $x \in D^+(S)$*, then* $I^-(x) \cap J^+(S) \subseteq D^+(S)$*;*

(4.) If S is closed, $\partial D^+(S) = H^+(S) \cup S$*;*

(5.) If S is closed, $\partial D(S) = H(S)$.

Next, we will show that Cauchy Surface S must be connected.

Proposition 3.3.6. *A Cauchy surface S in M must be connected.*

Proof. There exists a smooth timelike vector field on M. We have a smooth family of integrated future timelike curves λ induced from the vector field. There is a map $T : M \to S$ such that p maps to $\lambda(-\infty, \infty) \cap S$ where λ cuts p. The map is well-defined as S is a Cauchy surface. Since M is connected, it suffices to show T is continuous. Let d be the natural distance function between p and $q \in M$ with respect to a Riemannian metric on M. For any $q \in M$, any sequence q_n converges to q, we let $T(q) = \lambda_q(t_q)$ and $T(q_n) = \lambda_{q_n}(t_{q_n}) \in S$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $d(\lambda_q(t), \lambda_q(t_q)) < \epsilon$ for $t \in [t_q - \delta, t_q + \delta]$. By smoothness of ODE theorem, there exists N such that $d(\lambda_{q_n}(t), \lambda_q(t)) < \epsilon$ for $t \in [t_q - \delta, t_q + \delta]$ and $n \geq N$. Since, $\lambda_q(t_q - \delta)$ lies in $I^-(S)$ while $\lambda_q(t_q + \delta)$ lies in $I^+(S)$. We can make N larger such that $\lambda_{q_n}(t_q - \delta) \in I^-(S)$ and $\lambda_{q_n}(t_q + \delta) \in I^+(S)$ for all $n \geq N$. Since S is Cauchy surface, we have $\lambda_{q_n}(t_{q_n}) \in (t_q - \delta, t_q + \delta)$ for $n \geq N$. As a result, for $n \geq N$,

$$\begin{aligned} d(T(q_n), T(q)) &= d(\lambda_{q_n}(t_{q_n}), \lambda_q(t_q)) \\ &\leq d(\lambda_{q_n}(t_{q_n}, \lambda_q(t_{q_n})) + d(\lambda_q(t_{q_n}, \lambda_q(t_q)) \\ &\leq 2\epsilon. \end{aligned}$$

□

Finally, we will show that if M has a Cauchy surface S, then M is globally hyperbolic which is the main part in the chapter.

Lemma 3.3.7. *If S is achronal and $x \in D^\pm(S) - H^\pm(S)$, then every endless causal curve with future end-point x meets $S - H^\pm(S)$ and contains a point in $I^\mp(S)$.*

Proof. I just show the case of $x \in D^+(S) - H^+(S)$. There exist $y \in I^+(x) \cap D^+(S)$. For any past endless causal curve α with $\alpha(0) = x$, by the Lemma 3.3.3 and the definition of $H^+(S)$, there exists a endless timelike curve β with $\beta(0) = y$, $\beta(t) \in I^+(\alpha(t))$and $d(\gamma(t), \beta(t)) < \frac{4}{1+t}$. Then $\beta \cap S \neq \emptyset$. So there exists $t_0 > 0$ such that $\beta(t_0) \in S$. As a result, $\alpha(t_0) \in I^-(S)$. Moreover, $\alpha \cap S \neq \emptyset$. Let w be an intersection point. We have $y \in I^+(w)$ which says that $w \notin H^+(S)$ [by the definition of $H^+(S)$]. □

Proposition 3.3.8. *If S is achronal and closed, then*

(1.) int$(D(S))$ is strongly casual,

(2.) and u, $v \in$ int$(D(S))$, $J^+(u) \cap J^-(v)$ is compact.

Proof. To show (1.), let $V = \cup_{\{x \in M\}} I^+(x) \cap I^-(x)$. Clearly, $V \cap D(S) = \emptyset$. For any $z \in \mathrm{int}(D(S)) \cap \partial V$, there exists a sequence $\{v_n\} \in V$ such that $v_n \to z$. Hence, $v_n \in \mathrm{int}(D(S))$ for large n. Contradiction. So, $\mathrm{int}(D(S)) \cap \partial V = \emptyset$, too. Suppose there exists $p \in \mathrm{int}(D(S))$ at which strong casuality fails. By the Theorem 3.2.9, the conditions (a), (b), (c) and (d) are rejected since $p \notin \partial V$. For the conditions (e), we let γ be a null geodesic from p along which strong causality fails. From the item 5 of the Proposition 3.3.5, p is in at least one of the following three sets: $D^+(S) - H^+(S) - S$, $D^-(S) - H^-(S) - S$ or $S - H(S)$. In the first case, by the Lemma 3.3.7, γ has some point $q \in I^-(S)$. Then $q \preceq p$ and $q \neq p$. As $q \in I^-(S)$ and $p \in I^+(S)$, there exist y, $x \in S$ such that $y \ll p$ and $q \ll x$. As a result, $y \ll x$. It contradicts with achronal S. We have the same conclusion in the second case. In the third case, there exists q_1, $q_2 \in \gamma$ such that $q_1 \in I^-(S)$ and $q_2 \in I^+(S)$. Clearly, $q_1 \preceq q_2$. If $q_1 = q_2$, it contradicts with achronal S. If $q_1 \neq q_2$, by the Lemma 3.2.6 and 3.3.7, it also contradicts with achronal S.

To show (2.), suppose there exist $p, q \in \mathrm{int}(D(S))$ such that $J^+(p) \cap J^-(q)$ is non-compact. It means there exist a sequence of $a_n \in J^+(p) \cap J^-(q)$ which has no converging subsequence. W.L.O.G., we can assume $a_n \in D^-(S)$ for all n. Let γ_n be a future causal curve form $p = \gamma(0)$ through a_n to q while γ_n is piecewise differentiable and arc-length parameterized with respect to a complete Riemanian metric. We extend γ_n to be future endless. By the limit curve Theorem 3.1.3, there exists a future endless causal curve γ to which γ_n converge locally uniformly and $\gamma(0) = p$. We take $a_n = \gamma(t_n)$ with $t_n > 0$. We claim $t_n \to +\infty$. Suppose t_n is bounded. W.L.O.G., we can assume t_n converges to s. As a result, $a_n \to \gamma(s)$. It contradicts with the assumption of $\{a_n\}$. The claim is done. From the item 5 of the Proposition 3.3.5, $p \in D^-(S) - H^-(S)$. By the Lemma 3.3.7, we have $\gamma(t_0) \in I^+(S)$ for some $t_0 > 0$. Hence, $\gamma(t) \in I^+(S)$ for $t > t_0$. As a result, $a_n \in I^+(S)$ for large n. $a_n \in D^-(S) \cap I^+(S)$ means S is not achronal. Contradiction. $\square$

As a result, we have the following theorem

Theorem 3.3.9. *The existence of a Cauchy Surface $S \subseteq M$ implies M is globally hyperbolic.*

Proof. If M has a Cauchy surface S, then $\text{int}(D(S)) = M$. By the Proposition 3.3.9, M is globally hyperbolic. $\square$

In fact, in [3], Geroch R.P. showed that the converse is true.

Chapter 4

Conjugate Points

In the section 4.1, we will discuss the space of causal curves and its topology. Also we will show that the length function on this space is upper semi-continuous and the maximal curve is a causal geodesic in some conditions. In section 4.2, we will introduce Jacobi field and conjugate points in a space-time manifold. We will study the relationship between the length of a causal geodesic and its first conjugate point. We will show that the length of a timelike geodesic is not maximal after a conjugate point. Also, let $\gamma : [0, t] \to M$ be a future null geodesic. we will show there is a future timelike curve from $\gamma(0)$ to $\gamma(t)$ arbitrarily close to γ if there is a conjugate point $\gamma(t_0)$ where $t_0 \in (0, t)$. In the section 4.3, we will derive Raychauduri equation and use it to study the conditions for the existence of conjugate points.

4.1 Space of causal curves

Let K be the subset of M consisting of all points at which M is strongly casual (Please refer to p.9 in Chapter 2). By the Proposition 3.2.5, K is open. Let C be a subset of K and let A and B be subsets of C. We define $\Im_C(A, B) = \{\gamma|\gamma$

is a causal curve lying in C from a point of A to a point of B.}.

The topology on $\Im_K(K, K)$ induced by a base for open sets in $\Im_K(K, K)$ which is $\Im_R(P, Q)$ where P, Q and R are open in M with $P, Q \subseteq R$ and $R \subseteq K$. We fix p and $q \in K$ with $p \ll_K q$. Since K is strongly casual, we can choose a connected open set A of p, B of q such that all causal curve from A to B are future.

Theorem 4.1.1. *If C is an open and path-connected with $C \subset\subset K$ and C is compact in K and A and B are closed subsets of $\overline{C}$, then $\Im_{\overline{C}}(A, B)$ is compact.*

Proof. Since the topology of $\Im_K(K, K)$ is composed of a countable basis. It suffices to show that every converging sequence $\{\gamma_n\} \subseteq \Im_{\overline{C}}(A, B)$ has a converging subsequence in $\Im_{\overline{C}}(A, B)$.

Let h be a complete Riemannian metric on M. We consider $\gamma_n : [0, b_n] \to C$ is finite piecewise unions of causal geodesic from a point in A to a point in B and is parametized by arclength with respect to h. We let $r(\gamma_n) = \int_0^{b_n} \sqrt{h(\gamma', \gamma')}dt$.

We first show that there exists $H > 0$ with $r(\gamma_n) \leq H$ for all n. We cover C with causally convex neighbourhood U_α such that

(1.) $U_\alpha \subset\subset N_\alpha$ where N_α is a convex normal neighbourhood;

(2.) U_α has a compact closure such that it is contained in one coordinate chart;

(3.) There exists a $G_\alpha > 0$ such that for any timelike curve lying in U_α, it is also a timelike curve with respect to $-G_\alpha dt^2 + dx_1^2 + dx_2^2 + dx_3^2$.

Following the proof in the Theorem 3.1.3, there exists $H_\alpha > 0$ such that $r(\gamma_n|_{U_\alpha}) < H_\alpha$ for all n. Since $\overline{C}$ is compact, it is covered with finitely many of U_α. Hence, there exists $H > 0$ such that $r(\gamma_n) < H$ for all n. Hence, $b_n < H$ for all n.

Next, we will show that γ_n has a converging subsequence converges γ with $\gamma \in \Im_{\overline{C}}(A, B)$ in sense of the topology on $\Im$. Since both A and B are compact sets and $\gamma_n(0) \in A$. W.L.O.G, we can assume $\gamma_n(0)$ converges to $p \in A$ and γ_n is a future causal curve. We extends γ_n to be future endless. By the Theorem 3.1.3, there exists a future endless causal curve γ such that γ_n converges to it locally uniformly with $\gamma(0) = p$. Since $b_n \in [0, H]$, we can assume b_n converges to b. Then, $\gamma_n(b_n)$ converges to $\gamma(b)$. $\gamma(b_n) \in B$ which is closed implies $\gamma(b) \in B$. As a result, $\gamma|_{[0,b]} \in \Im_{\overline{C}}(A, B)$ and $\gamma_n|_{[0,b_n]}$ converges to $\gamma|_{[0,b]}$ in the sense of the topology induced on $\Im_K(K, K)$.

Finally, for any future causal curve $\{\gamma_n|_{[0,b_n]}\} \subseteq \Im_{\overline{C}}(A, B)$. There exists a piecewise union of future causal curve $\widetilde{\gamma_n} \in \Im_{\overline{C}}(A, B)$ such that $d_0(\widetilde{\gamma_n}(t), \gamma_n(t)) < \frac{1}{n}$ for $t \in [0, b_n]$ where d_0 is mentioned in the Theorem 3.1.3. $\widetilde{\gamma_n}$ converges to γ in the sense of the topology on $\Im_K(K, K)$ implies γ_n converges to γ in the same topology. $\square$

Corollary 4.1.2. *Let S be closed and achronal. Suppose strong causality holds at each point of S. Let $y, z \in int(D(S))$. Then, $\Im(\{y\}, \{z\})$ is compact.*

Next, we discuss the definition of length function on $\Im_K(A, B)$. Let $\gamma : [a, b] \to M$ be a piecewise C^1 causal curve. It is natural to define the length l of γ as $l(\gamma) = \int_a^b \sqrt{-g(\gamma', \gamma')}dt$ where g is a Lorentzian metric. Also, when we consider the relationship between maximal curve and geodesic, by p. 53-54 in [9], we have the following proposition.

Proposition 4.1.3. *Let U be a convex normal neighbourhood and let $p, q \in U$ which p and q can be joint by a future causal curve lying in U. Then if $\widehat{pq}$ is a causal geodesic lying in U, then $l(pq) > l(\gamma)$ where γ is any piecewise differentiable curve lying in U from p to q.*

We want to extend the length function l to all C^0 curve, so we define the length function $L : \Im_K(A, B) \to [0, \infty)$ as follows:

$$L(\gamma) = \inf_{\{\gamma \in \text{ open } C_R(P,Q)\}} \sup_{\{\lambda \in C_R(P,Q) \text{ is } C^{1-} \text{ curve}\}} l(\lambda).$$

By the same method in the second paragraph of the Proposition 4.1.1, it is easy to show that $L(\gamma)$ is finite for any $\gamma \in \Im_K(A,B)$. Also, we want to show that $L(\gamma) = l(\gamma)$ if γ is a piecewise differentiable curve in $F_K(K,K)$. First, we define an orthonormal basis along a timelike geodesic.

Definition 4.1.4. *Let γ be a future timelike geodesic with $\langle\gamma',\gamma'\rangle = -1$. At $\gamma(0)$, there exists spacelike vectors e_1, e_2 and e_3 such that $\langle\gamma',e_i\rangle = 0$ and*

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases}$$

We parallel transport $\{e_1, e_2, e_3, \gamma'\}$ along γ. Then, $\{e_1(t), e_2(t), e_3(t), \gamma'(t)\}$ is an orthonormal basis along γ.

Lemma 4.1.5. *l is upper semi-continuous on all piecewise differentiable curve in $\Im_K(A,B)$.*

Proof. For any C^{1-} γ in $\Im_K(A,B)$, first, we parameterize γ by arc-length with respect to a lorentzian metric g and assume γ is future and lies in a convex normal neighbourhood U. Let $\{\gamma'(t), e_1(t), e_2(t), e_3(t)\}$ be an orthonormal basis along γ. Hence, $H(x_1,x_2,x_3,s) = \exp_{\gamma(s)}(x_1e_1 + x_2e_2 + x_3e_3) : O \subseteq R^4 \to H(O) \subseteq U$ is a coordinate chart where $H(O)$ is an open set of γ. We consider all causal curve lying in U only. By the Proposition 3.2.3, H(O) is a causally convex.

Then, in a local coordinate $\{\gamma'(t), e_1(t), e_2(t), e_3(t)\}$, for a metric along $\gamma(t)$, $g = \left[\begin{array}{c|c} -1 & \vec{0} \\ \hline \vec{0} & (r_{ij}) \end{array}\right]$ where (r_{ij}) is a 3×3 symmetric matrix which is positive definite which is an identity along γ. For any small $\epsilon > 0$, any $s \in [0, l(\gamma)]$, there exists $V \subseteq H(O)$ where $\gamma(s) \in V$ such that for all $q \in V$

(1.) $g_{00} > -1 - \epsilon$;

(2.) $-\epsilon < g_{i0}(q) < \epsilon$;

(3.) $r_{ij}(q)x'_ix'_j > \epsilon\sum_{i=1}^{3} x_i'^2$;

(4.) $(r_{ij})(q)$ is positive definite.

Also, in a local coordinate of H, $\frac{\partial}{\partial s}|_{\gamma(a)} = \gamma'(a)$ and $\frac{\partial}{\partial s}|_{\gamma(b)} = \gamma'(b)$ are timelike vectors. Hence, there exists open $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $\frac{\partial}{\partial s}$ is future timelike on A_1 and B_1 and any causal curve from A_1 to B_1 lying in V must be future.

For any future casual curve $\rho \in \Im_V(A_1, B_1)$, $\rho(t) = H(x_1(t), x_2(t), x_3(t), s(t))$. Hence, $\rho'(t) = \sum_{i=1}^{3} x'_i\frac{\partial}{\partial x_i} + s'(t)\frac{\partial}{\partial s}$. Also, by the item 4., $s'(t) = 0$ at some t_0 implies $x'_i(t_0) = 0$ for $i = 1, 2, 3$. However, it contradicts with the fact that $\rho'(t_0)$ is future causal. Hence, we can reparameterize $\rho(t)$ by s. It means $\rho(s) = H(x_1(s), x_2(s), x_3(s), s)$ where s is from a' to b'.

$$
\begin{aligned}
l(\rho|_{[a',b']}) &= \int_{a'}^{b'} [-g(\rho'(s), \rho'(s))]^{\frac{1}{2}} ds \\
&= \int_{a'}^{b'} [-g_{00} - 2\sum_{i=1}^{3} g_{i0}x'_i - \sum_{i,j=1}^{3} r_{ij}x'_ix'_j]^{\frac{1}{2}} ds \\
&\leq \int_{a'}^{b'} [-g_{00} + 2\sum_{i=1}^{3} |g_{i0}x'_i| - \epsilon\sum_{i=1}^{3} x_i'^2]^{\frac{1}{2}} ds \qquad \text{By (3.)} \\
&\leq \int_{a'}^{b'} [-g_{00} + 2\sum_{i=1}^{3} \epsilon|x'_i| - \epsilon\sum_{i=1}^{3} x_i'^2]^{\frac{1}{2}} ds \qquad \text{By (2.)} \\
&\leq \int_{a'}^{b'} [-g_{00} + 2(\sum_{i=1}^{3} \epsilon|x'_i|^2)^{\frac{1}{2}}(\sum_{i=1}^{3} \epsilon)^{\frac{1}{2}} - \epsilon\sum_{i=1}^{3} x_i'^2]^{\frac{1}{2}} ds \text{ By Cauchy Schwarz's inequality} \\
&\leq \int_{a'}^{b'} [-g_{00} + \sum_{i=1}^{3} \epsilon|x'_i|^2 + \sum_{i=1}^{3} \epsilon - \epsilon\sum_{i=1}^{3} x_i'^2]^{\frac{1}{2}} ds \qquad \text{By AM}\geq\text{GM} \\
&\leq \int_{a'}^{b'} [1 + 4\epsilon]^{\frac{1}{2}} ds \qquad \text{By (1.)} \\
&= [1 + 4\epsilon]^{\frac{1}{2}} |a' - b'|.
\end{aligned}
$$

Since $\rho(a') \in A_1$ and $\rho(b') \in B_1$ where a' and b' are s-coordinate of $\rho(a')$ and

$\rho(b')$ respectively. Hence a' and b' are very near to 0 and $l(\gamma|_{[a,b]})$ respectively. We have, $l(\rho) \leq l(\gamma) + O(\epsilon)$. Thus, l is upper semi-continuous.

If the whole γ does not lies inside a single convex normal neighbourhood, then it is covered with a finite number of convex normal neighbourhood. By the above method, we can also conclude l is upper semi-continuous. □

Proposition 4.1.6. *$L(\gamma) = l(\gamma)$ if γ is piecewise and in $\Im_K(A, B)$*

Proof. On one hand, by definition of L, $L(\gamma) \geq l(\gamma)$. On the other hand, since l is upper semi-continuous, for any $\epsilon > 0$, there exist X, Y and O such that $l(\rho) \leq l(\gamma) + \epsilon$ for all $\rho \in \Im_O(A, B)$ and ρ is piecewise differentiable. Then, $L(\gamma) \leq l(\gamma) + \epsilon$ for all $\epsilon > 0$. Hence, $l(\gamma) \geq L(\gamma)$. □

As a result, by the first paragraph on p.31 and the Propositions 4.1.6, we can say L is reasonably defined. By the definition of L, we have

Theorem 4.1.7. *L is upper semi-continuous on $\Im_K(A, B)$.*

If S is achronal, the property of L can be used to study the relationship between maximal length and causal geodesic in $\mathrm{int}(D(S))$.

Corollary 4.1.8. *If S is achronal and $p, q \in$ int($D(S)$) with $p \preceq q$, then causal geodesic $\widehat{pq}$ from p to q exists and $L(\widehat{pq}) \geq L(\gamma)$ where γ is any causal curve from p to q.*

Proof. By the Theorem 3.3.8, we have $J^+(p) \cap J^-(q)$ is compact. By the Corollary 4.1.2 and the Theorem 4.1.7, there exists a future causal curve γ from p to q whose length is maximal. It is easy to show that $J^+(p) \cap J^-(q) \subseteq \mathrm{int}(D(S))$. Hence, $\gamma \subseteq \mathrm{int}(D(S))$.

First, we consider $p \ll q$. For any $\gamma|_{[a,b]} \in \Im_{J^+(p) \cap J^-(q)}(p, q)$, it is covered with $\{U(t)|U(t)$ is causally convex, $\gamma(t) \in U(t)$ and $U(t) \subseteq N(t) \subseteq J^+(p) \cap J^-(q)$ for some convex normal neighboruhood $N(t)\}$. It is covered with a finite number of $U(t_1)$,$U(t_2)$,...,$U(t_n)$. By the Theorem 3.1.1, there exists i such that $\gamma|_{U(t_i)}$ is not a single null geodesic. Let say $i = 1$.

We claim $\gamma|_{U(t_1)}$ is a timelike geodesic. Suppose it is not a timelike geodesic. Let $\gamma[0,1] \subseteq U(t_1)$ with $\gamma(0)$ and $\gamma(1) \in \partial U(t_1)$. For simplicity, we denote $\widehat{ab}$ is a future geodesic lying in $U(t_1)$ from a to b. By the assumption of $\gamma|_{U(t_1)}$, there exists b in $\gamma[0,1]$ but not in $\widehat{\gamma(0)\gamma(1)}$. There exists an arbitrarily small open neighbourhood O of b such that $O \cap \widehat{\gamma(0)\gamma(1)} = \emptyset$. By the Proposition 4.1.3, there exists $\epsilon > 0$ such that $l(\widehat{\gamma(0)b}) + l(\widehat{b\gamma(1)}) < l(\widehat{\gamma(0)\gamma(1)}) - 3\epsilon$. Indeed, l is upper semi-continuous, there exists a neighbourhood $\Im_{V_1}(\gamma(0), O)$ and $\Im_{V_2}(O, \gamma(1))$ of $\widehat{\gamma(0)b}$ and $\widehat{b\gamma(1)}$ respectively such that $l(\alpha_1) < l(\widehat{\gamma(0)b}) + \epsilon$ and $l(\alpha_2) < l(\widehat{b\gamma(1)}) + \epsilon$ for piecewise differentiable $\alpha_1 \in \Im_{V_1}(\gamma(0), O)$ and $\alpha_2 \in \Im_{V_2}(O, \gamma(1))$. Hence, for any piecewise differentiable future causal curve α from $\gamma(0)$ to $\gamma(1)$ lying in $V_1 \cup V_2$ with $\alpha(\frac{1}{2}) \in O$, $l(\alpha) < l(\widehat{\gamma(0)b}) + l(\widehat{b\gamma(1)}) + 2\epsilon < l(\widehat{\gamma(0)\gamma(1)}) - \epsilon$. Hence, $L(\gamma|_{U(t_1)}) < l(\widehat{\gamma(0)\gamma(1)})$. Contradiction since $L(\gamma[0,1])$ is maximal among all future causal curve lying in $U(t_1)$ from $\gamma(0)$ to $\gamma(1)$.

Then, for $\gamma|_{U(t_2)}$, by a similar argument, we can also conclude that it is a future timelike geodesic. By induction, we can conclude γ is a future timelike geodesic.

Next, we consider $p \preceq q$ and $p \not<< q$. By the Theorem 3.1.1, p and q can be joint by a single null geodesic only. Hence, γ must be a single null geodesic and $L(\gamma) = l(\gamma) = 0$. $\square$

4.2 Jacobi field, conjugate point and length of geodesic

First, we define a Jacobi field in space-time.

Definition 4.2.1. *Let $\Sigma \subseteq M$ be a smooth spacelike submanifold. Let $\gamma : [0, b] \to$*

M be a future timelike geodesic which is affine parametized with $\langle \gamma', \gamma' \rangle = -1$ and meets Σ, orthogonally at $\gamma(0)$. $F : [0, b] \times (-\epsilon, \epsilon) \to M$ be a variation of γ such that for every $t \in (-\epsilon, \epsilon)$, $F_u(t) = F(t, u)$ is a future timelike geodesic with $\gamma(t) = F_0(t)$ and $F_u(0) \in \Sigma$. Then, the variational vector field $J(t) = \frac{\partial F}{\partial t}|_{(t,0)}$ is a Jacobi field along γ in a timelike case. Similarly, if γ and $F_u(t)$ are future null geodesics and are parametrized by an affine parameter, then $J(t) = \frac{\partial F}{\partial u}|_{(t,0)}$ is a Jacobi field along γ in a null-like case.

We should note that Σ can be a point. Also, referring to p.224-227 in [5], there is a equivalent definition of the Jacobi field which is viewed as an ODE with some initial conditions.

Theorem 4.2.2. *$J(t)$ is a Jacobi field along γ in a timelike case if and only if*

(1.) $\nabla_{\gamma'} \nabla_{\gamma'} J(t) + R(J, \gamma')\gamma' = 0$ *along* γ*;*

(2.) $J(0) \in T_{\gamma(0)}\Sigma$*;*

(3.) for all $v \in T_{\gamma(0)}\Sigma$*, we have*$\langle \nabla_{\gamma'(0)} J(0), v \rangle$ $+\langle \nabla_{J(0)} \widetilde{v}, \gamma'(0) \rangle = 0$ *where* $\widetilde{v}$ *is a tangent vector field on* Σ *around* $\gamma(0)$ *with* $\widetilde{v}|_{\gamma'} = v$*.*

$J(t)$ is a Jacobi field along γ in a null like case if and only if

(1.) $\nabla_{\gamma'} \nabla_{\gamma'} J(s) + R(J, \gamma')\gamma' = 0$ *along* γ*;*

(2.) $J(0) \in T_{\gamma(0)}\Sigma$*;*

(3.) for all $v \in T_{\gamma(0)}\Sigma$*, we have* $\langle \nabla_{\gamma'(0)} J(0), v) \rangle$ $+\langle \nabla_{J(0)} \widetilde{v}, \gamma'(0) \rangle = 0$ *where* $\widetilde{v}$ *is a tangent vector field on* Σ *around* $\gamma(0)$ *with* $\widetilde{v}|_{\gamma'} = v$*;*

(4.) $\langle J'(t), \gamma'(t) \rangle = 0$ *along* γ*.*

There is a remark in this theorem. Let $\widetilde{\gamma'}$ be an extension of $\gamma'(0)$ on Σ locally. Then, the condition 2 is equivalent to say that the projection of $(J' - \nabla_{J(0)} \widetilde{\gamma'})$ to $T_{\gamma(0)}\Sigma$ is equal to 0.

We can use an orthonormal basis mentioned in the Definition 4.1.4 to express a Jacobi field along a timelike geodesic. However, we cannot use the basis to express it along a null geodesic. We will introduce a pseudo-orthonormal basis to deal with it.

Definition 4.2.3. *Let γ be a null geodesic. At $\gamma(0)$, there exists a null vector n, spacelike vectors e_1 and e_2 such that we have $\langle n, n\rangle = 0$, $\langle \gamma', e_i\rangle = \langle n, e_i\rangle = 0$ and $\langle n, \gamma'\rangle = 1$ and*

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We parallel transport $\{e_1, e_2, n, \gamma'\}$ along γ. Then, $\{e_1(t), e_2(t), n(t), \gamma'(t)\}$ is a pseudo-orthonormal basis along γ.

Next, we define a conjugate point to Σ.

Definition 4.2.4. *Let γ be a future causal geodesic. If there exists a non-trivial Jacobi field along γ from Σ to q such that $J = 0$ at q, then q is a conjugate point to Σ.*

Proposition 4.2.5. *Let $\gamma : [0, b] \to M$ be a future timelike geodesic and $\{e_1, e_2, e_3, \gamma'\}$ be a orthonormal basis along γ. Then, we have*

(1.) Let Σ be a spacelike hypersurface to which γ is orthogonal at $\gamma(0)$ and $e_1(0)$, $e_2(0)$ and $e_3(0)$ are in $T_{\gamma(0)}\Sigma$. We also let J_i be a Jacobi field along γ with $J_i(0) = e_i$ and $J_i'(0) = \nabla_{e_i}\widetilde{\gamma}'|_\Sigma$ where $\widetilde{\gamma}'$ is any normal extension vector field of $\gamma'(0)$ on Σ. $\gamma(b)$ is a conjugate point to Σ along γ if and only if $\triangle = 0$ at $\gamma(b)$ where $\triangle$ is a volume element spanned by J_1, J_2, J_3 and γ'.

(2.) Let Σ be a point. We also let J_i be a Jacobi field along γ with $J_i(0) = 0$ and $J_i'(0) = e_i$. $\gamma(b)$ is a conjugate point to $\gamma(0)$ along γ if and only if $\triangle = 0$ at $\gamma(b)$ where $\triangle$ is a volume element spanned by J_1, J_2, J_3 and γ'.

Proof. We will prove the case (1.) only.

($\Rightarrow$) There is a non-trivial Jacobi field J along γ such that $J(b) = 0$. $J(0) = \sum_{i=1}^{3} a_i J_i(0)$ and $J'(0) = (\nabla_J \gamma')|_{t=0} = \sum_{i=1}^{3} a_i J_i'(0)$ where a_i are constant with some $a_i \neq 0$. $H(t) = \sum_{i=1}^{3} a_i J_i(t)$ is a Jaocbi field and $H(0) = J(0)$, $H'(0) = J'(0)$. By the uniqueness of ODE theorem, $J(t) = \sum_{i=1}^{3} a_i J_i(t)$. $J(b) = 0$ implies J_1, J_2 and J_3 are linear dependent at w. Hence, $\triangle = 0$ at $\gamma(b)$.

($\Leftarrow$) $\triangle = 0$ at $\gamma(b)$. There exists some a_i which is not all zero such that $\sum_{i=1}^{3} a_i J_i = 0$ at $\gamma(b)$. Let $J(t) = \sum_{i=1}^{3} a_i J_i(t)$. It is a Jacobi field along γ which has a conjugate point at $\gamma(b)$. $\square$

Remark: If $\langle \gamma', \gamma' \rangle = -1$, then we have $\triangle = \sqrt{\langle J_i, J_j \rangle}$.

Proposition 4.2.6. *Let $\gamma : [0, b] \to M$ be a future null geodesic and $\{e_1, e_2, n, \gamma'\}$ be a pseudo-orthonormal basis along γ. Then, we have*

(1.) Let Σ be a spacelike two-surface to which γ is orthogonal at $\gamma(0)$ and $e_1(0)$ and $e_2(0)$ are in $T_{\gamma(0)}\Sigma$. We also let J_i be a Jacobi field along γ with $J_i(0) = e_i$ and $J_i'(0) = \nabla_{e_i} \widetilde{\gamma'}|_\Sigma$ where $\widetilde{\gamma'}$ is any normal extension vector field of $\gamma'(0)$ on Σ. $\gamma(b)$ is a conjugate point to Σ along γ if and only if $\triangle = 0$ at $\gamma(b)$ where $\triangle = \sqrt{det(\langle J_i, J_j \rangle)}$ is a volume element spanned by J_1, J_2, n and γ'.

(2.) Let Σ be a point. We also let J_i be a Jacobi field along γ with $J_i(0) = 0$ and $J_i'(0) = e_i$. $\gamma(b)$ is a conjugate point to $\gamma(0)$ along γ if and only if $\triangle = 0$ at $\gamma(b)$ where $\triangle = \sqrt{det(\langle J_i, J_j \rangle)}$ is a volume element spanned by J_1, J_2, n and γ'.

Proof. We only prove the case (1.) only.

($\Rightarrow$) It is similar to the Proposition 4.2.5

($\Leftarrow$) $\triangle = 0$ at $\gamma(b)$. There exists some constant a_i and c which is not all zero such that $a_i J_i = c\gamma'$ at $\gamma(b)$. Let $J(t) = \sum_{i=1}^{3} a_i J_i(t) - \frac{ct}{b}\gamma'$. It is a Jacobi field along γ arising from Σ which has a conjugate point at $\gamma(b)$. $\square$

In order to study the relationship between the length of a timelike geodesic and its first conjugate point, we will briefly introduce a spacelike index form.

Let $\gamma : [0, b] \to M$ be a piecewise C^3 future timelike curve which is arc-length parameterized and is perpendicular to Σ at $\gamma(0)$. Let $\alpha : (-\epsilon, \epsilon) \times [0, b] \to M$ is a variation of $\gamma(s)$ such that

(1.) $\alpha(0, t) = \gamma(t)$;

(2.) there is a subdivision $0 = t_0 < t_1 < ... < t_n = b$ of $[0, b]$ such that α is C^3 on each $(\epsilon, \epsilon) \times [t_i, t_{i+1}]$;

(3.) $\alpha(u, 0) \in \Sigma$, $a(u, b) = \gamma(b)$;

(4.) for each fixed u, $\alpha_u(t) = \alpha(u, t)$ is a future timelike curve.

There are two remarks below.

(a.) For any α, we can reparametrize it such that $\dfrac{\partial \alpha}{\partial u}$ is orthogonal to $\gamma'(t)$ on $\gamma_{[0,b]}$. The length of a curve is invariant under parametrization, so , for simplicity, we assume $\dfrac{\partial \alpha}{\partial u}$ is orthogonal to $\gamma'(t)$ along the geodesic.

(b.) The condition 4 must be automatically satisfied if ϵ is small enough. $\gamma(s)$ itself is a timelike curve. Suppose for all $\epsilon > 0$, there exists $|u| < \epsilon$ such that $\alpha_u(t)$ is not a timelike curve. Hence, we can assume there exists a sequence $(u_n, t_n) \in (-\epsilon, \epsilon) \times [t_0, t_1]$ such that $u_n \to 0$ and $\langle \alpha'_{u_n}(t_n), \alpha'_{u_n}(t_n) \rangle \geq 0$. We can assume $t_n \to s$ for some $s \in [t_0, t_1]$. Hence, $\lim_{n\to\infty} \dfrac{\partial \alpha_{u_n}}{\partial t}(t_n) = \lim_{n\to\infty} \dfrac{\partial \alpha}{\partial t}(0, s)$ $= \gamma'(s)$. Contradiction as γ is a timelike curve.

$L(\alpha(u, [0, b])) = \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \sqrt{-\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle} dt$ is the length of a curve. It is a function of u. Then, we have the first variation formula

$$\frac{\partial L}{\partial u}|_{u=0} = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle \frac{\partial}{\partial t}, \frac{D}{\partial t}\frac{\partial}{\partial t} \rangle dt + \sum_{i=1}^{n-1} \langle \frac{\partial}{\partial u}, \triangle(\frac{\partial}{\partial t}) \rangle (t_i) + \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle (0)$$

where $\triangle\frac{\partial}{\partial t}(t_i) = \frac{\partial}{\partial t}\Big|_{t_{i+1}^-} - \frac{\partial}{\partial t}\Big|_{t_i^+}$. $\frac{\partial L}{\partial u}(0) = 0$ if γ is a timelike geodesic and it is orthogonal to Σ at $\gamma(0)$. Hence, a necessary condition for γ to be the longest curve from Σ to $\gamma(b)$ is that it must be a timelike geodesic which is orthogonal to Σ at $\gamma(0)$. To proceed further, we will study second derivative of L.

For the same γ as before. Let $\alpha : (\epsilon_1, \epsilon_1) \times (-\epsilon_2, \epsilon_2) \times [0, b] \to M$ from Σ to p such that

(1.) $\alpha(0, 0, t) = \gamma(t)$;

(2.) there is a subdivision $0 = t_0 < t_1 < ... < t_n = b$ of $[0, b]$ such that α is C^3 on each $(\epsilon_1, \epsilon_1) \times (\epsilon_2, \epsilon_2) \times [t_i, t_{i+1}]$;

(3.) $\alpha(u_1, u_2, 0) \in \Sigma$, $\alpha(u_1, u_2, b) = \gamma(b)$

(4.) for all constant u_1,u_2, $\alpha(u_1, u_2, t)$ is a future timelike curve;

(5.) $\frac{\partial}{\partial u_1}$ is smooth along $\frac{\partial}{\partial u_2}$ at $\gamma(t)$, vice versa.

Similarly, we can assume $\frac{\partial \alpha}{\partial u_1}$ and $\frac{\partial \alpha}{\partial u_2}$ is orthogonal to γ' along γ for simplicity. Then, we have the second variation formula

$$\frac{\partial^2 L}{\partial u_2 \partial u_1}\Big|_{u_1=u_2=0} = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle \frac{\partial}{\partial u_1}, \nabla_{\gamma'} \nabla_{\gamma'} \frac{\partial}{\partial u_2} - R(\frac{\partial}{\partial u_2}, \gamma')\gamma' \rangle (t) dt +$$
$$\sum_{i=1}^{n-1} \langle \frac{\partial}{\partial u_1}, \triangle[\nabla_{\gamma'}(\frac{\partial}{\partial u_2}]\rangle(t_i) - \langle \frac{\partial}{\partial u_1}, \nabla_{\frac{\partial}{\partial u_2}} T\rangle|_\Sigma + \langle \frac{\partial}{\partial u_1}, \nabla_{\gamma'} \frac{\partial}{\partial u_2}\rangle|_\Sigma$$

where T is a unit normal vector field on Σ such that $T|_{\gamma(0)} = \gamma'(0)$. Also, by direct computation,we can show that $\frac{\partial^2 L}{\partial u_2 \partial u_1}(0, 0, t) = \frac{\partial^2 L}{\partial u_1 \partial u_2}(0, 0, t)$.

Definition 4.2.7 (Space-like Hypersurface Index Form I_Σ)**.** *Let $\gamma : [0, b] \to M$ be a future timelike geodesic which is orthogonal to Σ at $\gamma(0)$ and is arc-length parameterized. Let Z_1 and Z_2 be piecewise smooth vector fields along γ such that*

1. *Z_1 and Z_2 are orthogonal to γ' along γ;*

2. $Z_1 = Z_2 = 0$ *at* $\gamma(b)$.

then

$$I_\Sigma(Z_1, Z_2) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle Z_1, \nabla_{\gamma'}\nabla_{\gamma'}(Z_2)\gamma' + R(Z_2, \gamma')\gamma'\rangle(t)dt +$$
$$\sum_{i=1}^{n-1} \langle Z_1, \triangle[\nabla_{\gamma'} Z_2](t_i) - \frac{\partial}{\partial t}\rangle)^{\frac{1}{2}} dt + \langle Z_1, \nabla_{\gamma'} Z_2\rangle|_\Sigma$$

where T *is a normal vector field on* Σ *with* $T|_{\gamma'(0)} = \gamma'(0)$

It is easy to show that I_Σ is symmetric bilinear form. Also, since for each variation of curve $\alpha(u,t)$, $L(\alpha_u([0,b])) = L(\gamma|_{[0,b]}) + [\frac{\partial L}{\partial u}(0)]u + [\frac{\partial^2 L}{\partial u^2}(0)]\frac{u^2}{2} + O(u^3)$ $= L(\gamma|_{[0,b]}) + I_\Sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) + O(u^3)$. If we show that $I_\Sigma(Z, Z) > 0$ for some of the above Z, then the length of $\gamma[0, b]$ is not locally maximal.

Theorem 4.2.8. *Let* $\gamma(t) : [0, b] \to M$ *be a future timelike geodesic which is perpendicular to a spacelike hypersurface* Σ *at* $\gamma(0)$*. If there is a conjugate point* $\gamma(t_0)$ *to* Σ *where* $0 < t_0 < b$ *along* $\gamma[0, b]$*, then the length of* γ *is not maximal.*

Proof. Suppose $\gamma(t_0)$ is a point conjugate to Σ along γ. Then we will claim the length of the geodesic $\gamma[0, b]$ is not maximal for any $b \geq t_0$. Let Y be a non-trivial Jacobi field along γ with $Y(0) \in T_{\gamma(0)}\Sigma$ and $Y(t_0) = 0$. Then we set

$$\widetilde{Y}(t) = \begin{cases} Y(t) & \text{if } t \leq t_0 \\ 0 & \text{if } t_0 < t \leq b \end{cases}$$

We let $W(t)$ be the parallel vector field along γ such that $W(t_0) = Y'(t_0) \neq 0$. Then we set $Y_\epsilon(t) = \phi(t)W(t) + \epsilon\widetilde{Y}(t)$ where ϕ is a smooth function such that $\phi(0) = \phi(b) = 0$ and $\phi(t_0) = -1$. It can be shown that Y is orthogonal to γ' for $t \in [0, b]$. Hence, $I_\Sigma(Y_\epsilon, Y_\epsilon) = 2\epsilon\langle Y(t_0), Y(t_0)\rangle + \epsilon^2 I_\Sigma(\widetilde{Y}, \widetilde{Y})$. $Y'(t_0) \neq 0$ is orthogonal to γ' which means $Y'(t_0)$ is a space-like vector. $\langle Y(t_0), Y(t_0)\rangle > 0$. Also, since Y is smooth, $I_\Sigma(Y, Y)$ is finite. As a result, if $\epsilon > 0$ is small enough, $I_\Sigma(Y_\epsilon, Y_\epsilon) > 0$.

To finish the proof, we need to construct a variation $\alpha(u,t)$ of γ from Σ to $\gamma(b)$ with $\frac{\partial\alpha}{\partial u}|_{u=0} = Y_\epsilon$. First, we assume γ is a unit timelike geodesic and extends it on $(-\zeta, b+\zeta)$ where $\zeta > 0$ is very small. Then, we can construct a vector field V on an open neighbourhood $U \supseteq \gamma(-\zeta, b+\zeta)$ such that $\langle V, V\rangle = -1$ and $V|_\gamma = \gamma'$. Next, there exists a coordinate map $F : (x_1, x_2, x_3, t) \to U$ such that $F(x_1, x_2, x_3, t)$ is an integral curve induced by the vector field V with the initial position $F(x_1, x_2, x_3, 0) \in \Sigma$. Since γ is orthogonal to Σ at $\gamma(0)$, it is easy to show that $\frac{\partial}{\partial x_i}$ is orthogonal to γ' along $\gamma[0,b]$. Hence, $Y_\epsilon(t) = \sum_{i=1}^{3} a_i(t)\frac{\partial}{\partial x_i}(t)$ where $a_i(b) = 0$ for $i = 1, 2, 3$. So, $\alpha(u,t) = F(a_1(t)s, a_2(t)s, a_3(t)s, t)$ is the required variation of γ. $\square$

Similarly, with slight modification of first and second variation formula, we also have the following theorem for Σ being a point.

Theorem 4.2.9. *Let $\gamma(t) : [0,b] \to M$ be a future timelike goedesic. If there is a conjugate point $\gamma(t_0)$ to $\gamma(0)$ where $0 < t_0 < b$ along $\gamma[0,b]$, then γ is not a maximal curve.*

Let us consider a conjugate point in a null-like case. Let γ be a future null geodesic which is orthogonal to a spacelike two-surface Σ. We will show that if there exists a conjugate point, $\gamma(t_0)$, to Σ along γ, then for any $t \geq t_0$, there exists a future timelike curve from Σ to $\gamma(t)$ which is arbitrarily close to $\gamma[0,t]$. Similarly, the result is still true if 'a spacelike-2 surface Σ' is replaced by 'a point', let say $\gamma(0)$.

Lemma 4.2.10. *Given $F(u,t) : (-\epsilon, \epsilon) \times [0,b] \to M$ which*

(1.) $F(0,t) = \gamma(t)$ is a null geodesic orthogonal to Σ at $\gamma(0)$;

(2.) $F(u,0) \in \Sigma$ and $F(u,b) = \gamma(b)$ for all $u \in (-\epsilon, \epsilon)$;

(3.) F is C^4 on $[-\epsilon, \epsilon] \times [0,b]$.

then if F satisfies that

(a.) $\frac{\partial}{\partial u}(0,t)$ *is orthogonal to* γ' *for* $t \in [0,b]$;

(b.) there exists $c > 0$ *such that for* $u = 0$ *and* $t \in [0,b]$, *we have*

$$\frac{d}{dt}[\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \gamma' \rangle + \langle \frac{\partial}{\partial u}, \nabla_{\gamma'} \frac{\partial}{\partial u} \rangle] - \langle \frac{\partial}{\partial u}, \nabla_{\gamma'} \nabla_{\gamma'} \frac{\partial}{\partial u} + R(\frac{\partial}{\partial u}, \gamma')\gamma' \rangle < -c < 0,$$

then there exists $\epsilon' < \epsilon$ *such that for each fixed* $u \in (-\epsilon', \epsilon')/\{0\}$, $F(u,t)$ *is a timelike curve.*

Proof. It suffices to show $\frac{d}{du}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle|_{u=0} = 0$ and $\frac{d^2}{du^2}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle|_{u=0} < -c$ for $t \in [0,b]$.

$$\frac{d}{du}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle|_{u=0} = 2\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle|_{u=0} = 2\frac{d}{dt}\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial t} \rangle|_{u=0} - 2\langle \frac{\partial}{\partial u}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} \rangle|_{u=0} = 0.$$

Also,

$$\begin{aligned}
&\frac{d^2}{du^2}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle|_{u=0} \\
&= [2\frac{d^2}{dudt}\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial t} \rangle - 2\frac{d}{du}\langle \frac{\partial}{\partial u}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} \rangle]|_{u=0} \\
&= [2\frac{d}{dt}(\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \frac{\partial}{\partial t} \rangle + \langle \frac{\partial}{\partial u}, \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial t} \rangle) - 2\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} \rangle - 2\langle \frac{\partial}{\partial u}, \nabla_{\frac{\partial}{\partial u}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} \rangle]|_{u=0} \\
&= [2\frac{d}{dt}(\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \frac{\partial}{\partial t} \rangle + \langle \frac{\partial}{\partial u}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial u} \rangle) - 2\langle \frac{\partial}{\partial u}, \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial u} + R(\frac{\partial}{\partial u}, \frac{\partial}{\partial t})\frac{\partial}{\partial t} \rangle]|_{u=0} \\
&= [2\frac{d}{dt}(\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \gamma' \rangle + \langle \frac{\partial}{\partial u}, \nabla_{\gamma'} \frac{\partial}{\partial u} \rangle) - 2\langle \frac{\partial}{\partial u}, \nabla_{\gamma'} \nabla_{\gamma'} \frac{\partial}{\partial u} + R(\frac{\partial}{\partial u}, \gamma')\gamma' \rangle]|_{u=0} \\
&< -c.
\end{aligned}$$

We have $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle(u,t) = \frac{1}{2}\frac{d^2}{du^2}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle|_{(0,t)} u^2 + \frac{1}{3!}\frac{d^3}{du^3}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle|_{\{\xi_u,t\}} u^3$ where $0 < \xi_u < u$. Since F is C^4 on $[-\epsilon, \epsilon] \times [0,b]$. There exists $d > 0$ such that $\frac{1}{3!}\frac{d^3}{du^3}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle|_{\{\xi_u,t\}} u^3 < du^3$ for $(u,t) \in [-\epsilon, \epsilon] \times [0,b]$. So, $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle(u,t) < -cu^2 + du^3$. As a result, we take $\epsilon' = \min\{\epsilon, \frac{c}{d}\}$. $F(u,t)$ is a timelike curve for $u \in (-\epsilon', \epsilon')/\{0\}$. □

Theorem 4.2.11. *Let $\gamma(t) : [0, b] \to M$ be a future null geodesic which is perpendicular to a spacelike-2 surface Σ at $\gamma(0)$. If $\gamma(t_0)$ is the first conjugate point $\gamma(t_0)$ to Σ where $0 < t_0 < b$ along $\gamma[0, b]$, then there is a variation of γ, $\alpha(u, t)$, such that $\alpha_u(t)$ gives a timelike curve from Σ to $\gamma(b)$ except $u = 0$.*

Proof. We divide the proof into three main parts.

(1.) We construct a variational vector field Z(t) and its acceleration vector field $A(t)$ along $\gamma[0, t_0 + \delta]$ where δ is arbitrarily small such that $Z(t) \perp \gamma'$ and there exist $c > 0$ such that

$$\frac{d}{dt}[\langle A(t), \gamma' \rangle + \langle Z(t), \nabla_{\gamma' Z(t)} \rangle] - \langle Z(t), \nabla_{\gamma'} \nabla_{\gamma'} Z(t) + R(Z(t), \gamma')\gamma' \rangle < -c.$$

(2.) We construct a variation $F : (-\epsilon, \epsilon) \times [0, t_0 + \delta] \to M$ such that

(a.) $\frac{\partial F}{\partial u}|_{u=0} = Z$,

(b.) $\nabla_{\frac{\partial}{\partial u}} \frac{\partial F}{\partial u}|_{u=0} = A$,

(c.) $F(u, 0) \in \Sigma$, $F(u, 1) = \gamma(t_0 + \delta)$, $F(0, t) = \gamma(t)$ and F is C^3.

With the Lemma 4.2.10, we have $F(u, t)$ is a timelike curve for $u \neq 0$ and $t \in [0, t_0 + \delta]$.

(3.) For any neighbourhood O of $\gamma[0, b]$, there exists a timelike curve lying inside O from Σ to $\gamma(b)$.

We prove the part 1. Let $\{e_1, e_2, n, \gamma'\}$ be a pseudo-orthonormal basis along γ with $e_1(0)$ and $e_2(0)$ are tangent vectors on Σ. Since $\gamma(t_0)$ is the first conjugate point. There exists a non-trivial Jacobi field J along $[0, b]$ such that $J(0) \in T_{\gamma'(0)}\Sigma$, $J(t_0) = 0$ and $J(t) = a_1(t)e_1 + a_2(t)e_2 + a_3\gamma'$. a_3 is identically zero on $[0, t_0]$. Otherwise, if $a_3(\eta) \neq 0$ for some $\eta \in [0, t_0]$, then $\widetilde{J}(t) = J(t) - a_3(\eta)\frac{t}{\eta}e_3$ is a Jacobi field such that $\widetilde{J}(\eta) = 0$. It contradicts with the first conjugate point $\gamma(t_0)$.

We claim there exists $\epsilon > 0$ such that there is no conjugate point on $\gamma[0, t_0 + \epsilon]$ except $t = t_0$ and $J(t) = f(t)W(t)$ on $\gamma[0, t_0 + \epsilon]$ where

(i.) $f(t)$ is smooth function and $W(t)$ is smooth vector field;

(ii.) $\langle W(t), W(t) \rangle = 1$;

(iii.)

$$f(t) \begin{cases} > 0 & \text{if } t \in [0, t_0) \\ = 0 & \text{if } t = t_0 \\ < 0 & \text{if } t \in (t_0, t_0 + \epsilon] \end{cases}$$

We let

$$f(t) = \begin{cases} \sqrt{\langle J(t), J(t) \rangle} = \sqrt{a_1^2 + a_2^2} & \text{if } t \in [0, t_0] \\ -\sqrt{\langle J(t), J(t) \rangle} = -\sqrt{a_1^2 + a_2^2} & \text{if } t \in [t_0, t_0 + \epsilon] \end{cases}.$$

$a_i(t) - a_i(t_0) = \int_0^1 \frac{d}{dz} a_i(z(t - t_0) + t_0)dz$. $a_i(t) - a_i(t_0) = \int_0^1 a_i'(z(t - t_0) + t_0)(t - t_0)dz = (t - t_0)\int_0^1 a_i'(z(t - t_0) + t_0)dz$. $\int_0^1 a_i'(z(t - t_0) + t_0)dz$ are smooth. For simplicity, we denote $h_i(t) = \int_0^1 a_i'(z(t - t_0) + t_0)dz$. Also, $J'(t_0) \neq 0$. There exists some i such that $a_i'(t_0) \neq 0$. Hence, $h_i(t_0) \neq 0$ for some i. It tells us that $f(t) = -(t - t_0)\sqrt{h_1^2 + h_2^2}$ is smooth on $[0, t_0 + \epsilon]$. Also, we let $W(t) = \dfrac{J(t)}{f(t)} = \displaystyle\sum_{i=1}^{2} \frac{-h_i(t)}{\sqrt{h_1(t)^2 + h_2(t)^2}} e_i + \frac{-h_3(t)}{\sqrt{h_1(t)^2 + h_2(t)^2}} \gamma'$. $W(t)$ is smooth on $[0, t_0 + \epsilon]$. Indeed, by $f'(t_0) < 0$ and the continuity of f', we can make $\epsilon > 0$ small such that $f(t) < 0$ for $t \in (t_0, t_0 + \epsilon]$

Next, we will construct $Z(t)$ by slightly stretching $W(t)$. We let $Z(t) = (\psi + f)W(t)$ where ψ is a C^2 function on $[0, t_0 + \epsilon]$. It is obvious that $Z(t) \perp \gamma'$ on $[0, t_0 + \epsilon]$. Also,

$$\nabla_{\gamma'}\nabla_{\gamma'} Z + R(Z, \gamma')\gamma = \psi'' W + 2\psi' \nabla_{\gamma'} W + \psi[\nabla_{\gamma'}\nabla_{\gamma'} W + R(W, \gamma')\gamma'].$$

Hence,

$$
\begin{aligned}
&\langle \nabla_{\gamma'}\nabla_{\gamma'}Z + R(Z,\gamma')\gamma, Z\rangle \\
&= \psi''(f+\psi) + 2\psi'(f+\psi)\langle \nabla_{\gamma'}W, W\rangle + (\psi+f)\psi\rangle\nabla_{\gamma'}\nabla_{\gamma'}W + R(W,\gamma')\gamma', W\rangle \\
&= (\psi+f)[\psi'' + \psi\langle \nabla_{\gamma'}\nabla_{\gamma'}W + R(W,\gamma')\gamma', W\rangle].
\end{aligned}
$$

We want the inner product strictly less than zero. First, we consider $\psi'' + \psi\langle \nabla_{\gamma'}\nabla_{\gamma'}W + R(W,\gamma')\gamma', W\rangle$. Since f, ψ and W are smooth. There exists $\alpha > 0$ such that $0 < \alpha^2 + \langle \nabla_{\gamma'}\nabla_{\gamma'}W + R(W,\gamma')\gamma', W\rangle$ for $t \in [0, t_0+\epsilon]$. Hence, $\psi'' + \psi\langle \nabla_{\gamma'}\nabla_{\gamma'}W + R(W,\gamma')\gamma', W\rangle > \psi'' - \alpha^2\psi$ which is an ODE inequality. We let $\psi(t) = \beta(e^{\alpha t} - 1)$ for some constant $\beta > 0$. We take $\beta = \frac{-f(t_0+\epsilon)}{e^{\alpha(t_0+\epsilon)-1}} > 0$ and locate the first zero of $(\psi + f)$ for $t \in [0, t_0+\epsilon]$. Since $(\psi+f)(t) > 0$ for $t \in [0, t_0]$ and $(\psi+f)(t_0+\epsilon) = 0$, by continuity of $\psi + f$, there exists a $\epsilon \geq \delta > 0$ such that $\psi + f$ is the first zero at $t_0 + \delta$. Hence, $Z(t) = [-\frac{f(t_0+\epsilon)}{e^{\alpha(t_0+\epsilon)}-1}(e^{\alpha t} - 1) + f(t)]W(t)$ on $[0, t_0+\delta]$.

Finally, we construct $A(t)$. Following the second paragraph of the Theorem 4.2.8, there exists a parametrization $\widetilde{X} : [0, t_0+\delta] \times (-\epsilon, \epsilon) \to M$ such that it satisfies the conditions (a) and (c) in the part (2). We let $A(t) = [\langle e_1, \nabla_{\frac{\partial \tilde{X}}{\partial u}} \frac{\partial \widetilde{X}}{\partial u}\rangle(0)\frac{t_0+\delta-t}{t_0+\delta}]e_1 + [\langle e_2, \nabla_{\frac{\partial \tilde{X}}{\partial u}} \frac{\partial \widetilde{X}}{\partial u}\rangle(0)\frac{t_0+\delta-t}{t_0+\delta}]e_2 + [\langle n, \nabla_{\frac{\partial \tilde{X}}{\partial u}} \frac{\partial \widetilde{X}}{\partial u}\rangle(0)\frac{t_0+\delta-t}{t_0+\delta}]\gamma' - [\langle Z, \nabla_{\gamma'}Z\rangle(t) + \{\langle \gamma', \nabla_{\frac{\partial \tilde{X}}{\partial u}} \frac{\partial \widetilde{X}}{\partial u}\rangle(0) + \langle Z, \nabla_{\gamma'}Z\rangle(0)\}\frac{t_0+\delta-t}{t_0+\delta}]n$. Then

$$
\begin{aligned}
&\frac{d}{dt}(\langle A(t), \gamma'\rangle + \langle \nabla_{\gamma'}Z, Z\rangle) \\
&= -\langle \gamma', \nabla_{\frac{\partial \tilde{X}}{\partial u}} \frac{\partial \widetilde{X}}{\partial u}\rangle(0) - \langle \nabla_{\gamma'}Z, Z\rangle(0)
\end{aligned}
$$

There exists a variational of null geodesic $H(u,t)$ such that $\frac{\partial H}{\partial u}|_{u=0} = J$ and $\frac{\partial H}{\partial t}|_{t=0}$ is normal to Σ.

$$
= -[\langle \gamma', \nabla_{\frac{\partial \tilde{X}}{\partial u}} \frac{\partial H}{\partial u}\rangle(0) + \langle \gamma', \nabla_{\frac{\partial \tilde{X}}{\partial u}} \frac{\partial \widetilde{X}}{\partial u} - \nabla_{\frac{\partial \tilde{X}}{\partial u}} \frac{\partial H}{\partial u}\rangle(0)] - \langle \nabla_{\gamma'}Z, Z\rangle(0)
$$

Since $\frac{\partial H}{\partial u}|_{(0,0)} = \frac{\partial \widetilde{X}}{\partial u}_{(0,0)} = J(0)$, we have $\langle \gamma', \nabla_J(\frac{\partial H}{\partial u} - \frac{\partial \widetilde{X}}{\partial u})\rangle = 0$.

$$
= -\langle \gamma', \nabla_J \frac{\partial H}{\partial u}\rangle(0) - \langle \nabla_{\gamma'}Z, Z\rangle(0)
$$

$$
\begin{aligned}
&= -\nabla_J \langle \gamma', \frac{\partial H}{\partial u} \rangle|_{\gamma(0)} + \langle \nabla_J \gamma', \frac{\partial H}{\partial u} \rangle|_{\gamma(0)} - \frac{1}{2}\nabla_{\gamma'} \langle Z, Z\rangle(0) \\
&= 0 + \langle Z', Z\rangle|_{\gamma(0)} - f(\psi' + f')(0) \\
&= -f\psi(0) \\
&< 0.
\end{aligned}
$$

We prove the part (2.). We let $V(t,u) = \exp^{-1}_{\gamma(t)}(\widetilde{X}(t,u))$ on $[0, t_0+\delta] \times (-\epsilon, \epsilon)$. For each $\gamma(t)$, there is a normal coordinate $(w_1, w_2, w_3, w_4) \to U$ where U is an open neighbourhood of $\gamma(t)$. In the coordinate, $A = A_1 \frac{\partial}{\partial w_1} + A_2 \frac{\partial}{\partial w_2} + A_3 \frac{\partial}{\partial w_3} + A_4 \frac{\partial}{\partial w_4}$ while $V(t,0) = V_1 \frac{\partial}{\partial w_1} + V_2 \frac{\partial}{\partial w_2} + V_3 \frac{\partial}{\partial w_3} + V_4 \frac{\partial}{\partial w_4}$. We let $B(t) = (A_1 - V_1)\frac{\partial}{\partial w_1} + (A_2 - V_2)\frac{\partial}{\partial w_2} + (A_3 - V_3)\frac{\partial}{\partial w_3} + (A_4 - V_4)\frac{\partial}{\partial w_4}$ be a vector field $\gamma[0, t_0+\delta]$. Then, we take $X : [0, t_0+\delta] \times (-\epsilon, \epsilon) \to M$ with $X(t,u) = \exp_{\gamma(t)}(V(t,u) + B(t)u^2)$ which satisfy the conditions (a), (b) and (c) in the part (2.).

We prove the part (3.). By the part (1.), (2.) and the Lemma 4.2.10, there exists a variation $F(u,t)$ of γ from Σ to $\gamma(t_0+\delta)$ such that $F_u(t)$ is a timelike curve except $u = 0$, then there exists $u \neq 0$ such that $F_u[0, t_0+\delta]$ lying in O. If s is near $t_0 + \delta$, then $F_u(s) \ll \gamma(b)$. There exists another future timelike curve $\beta(t)$ from $F_u(s)$ to $\gamma(b)$ lying in O. Hence, $F_u(t) \cup \beta(t)$ is a timelike curve from Σ to $\gamma(b)$ lying in O. $\square$

Similarly, we also have the following theorem for Σ being a point.

Theorem 4.2.12. *Let $\gamma(t) : [0, b] \to M$ be a future null geodesic. If $\gamma(t_0)$ is the first conjugate point $\gamma(t_0)$ to $\gamma(0)$ where $0 < t_0 < b$ along $\gamma[0, b]$, then there is a variation of γ, $\alpha(u,t)$, such that $\alpha_u(t)$ gives a timelike curve from $\gamma(0)$ to $\gamma(b)$ except $u = 0$.*

4.3 Congruence of causal geodesics

A congruence of timelike curves (γ) in M is a smooth family of timelike curves such that through each $p \in M$, there passes precisely one curve in this family.

If O is a sufficiently small compact region, one can represent a congruence by a diffeomorphism $f : [a, b] \times \Lambda \to O$ where $[a, b]$ is some closed interval of R^1 and Λ is a three dimensional manifold with boundary. f maps (t, x_1, x_2, x_3) to a point of the integral timelike curve $f_{(0,x_1,x_2,x_3)}(t)$ with initial value $(0, x_1, x_2, x_3)$ on Λ.

In this thesis, we usually consider a congruence of timelike geodesics which are affine parametrized with length $= -1$. [i.e. $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle = -1$ and $\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0$.]

Next, we will derives the Raychaudhuri equation in a timelike case.

Let T be a tangent of γ in (γ). For each point p, let $P_{ij} = g_{ij} + T_i T_j$ be the induced metric on the normal subspace of T. We consider $\nabla_i T_j$ and decompose it into symmetric and antisymmetric part. Since $T^i \nabla_i T_j = T^j \nabla_i T_j = 0$, $\nabla_i T_j$ is in the normal subspace. We can use P^{ij} to take trace. We now define three terms. Expansion $\theta = P^{ij} \nabla_i T_j = \nabla_i T^i = \text{div} T$. The shear $\sigma_{ij} = \nabla_{(i} T_{j)} - \frac{1}{3}\theta P_{ij}$. It is symmetric and traceless. The rotation $\omega_{ij} = \nabla_{[i} T_{j]}$. It is antisymmetric and traceless.

So, $\nabla_i T_j = \frac{1}{3}\theta P_{ij} + \sigma_{ij} + \omega_{ij}$.

When we consider the change of $\nabla_i T_j$ along the timelike geodesic.

$\nabla_T \nabla_i T_j = T^k \nabla_k \nabla_i T_j = T^k \nabla_i \nabla_k T_j + T^k R^l_{kij} T_l$
$= \nabla_i (T^k \nabla_k T_j) - (\nabla_i T^k)(\nabla_k T_j) + R_{kijl} T^l T^k = -(\nabla_i T^k)(\nabla_k T_j) - R_{ikjl} T^l T^k$.

Taking trace of $\nabla_T \nabla_i T_j$,we have

$\nabla_T \theta = -\frac{1}{3}\theta^2 - \sigma_{ij}\sigma^{ij} + \omega_{ij}\omega^{ij} - R_{ij} T^i T^j$. It is the Raychaudhuri equation in the timelike case.

More about the expansion θ, it is used to measure the average expansion of the infinitesimally nearby geodesic. In geometrical meaning, θ at p is a mean curvature of a surface which meet (γ) orthogonally around p. We will study the relationship between θ and conjugate point.

Let γ be a future timelike geodesic and $\{e_1, e_2, e_3, T\}$ be an orthonormal basis along γ mentioned in the Definition 4.1.4 where $T = \gamma'$. There is a relationship between $\triangle$ and θ where $\triangle$ is the volume element mentioned in the Proposition 4.2.5.

Proposition 4.3.1. *Let θ be an expansion of a congruence of timelike geodesics (γ). Then, $\theta = \frac{1}{\triangle}\nabla_{\gamma'}\triangle$ along a timelike geodesic.*

Proof. Let $\{e_1, e_2, e_3, T\}$ be a local basis. Then, $\triangle = \sqrt{-detg}$ is a volume element where g is the metric on M. $\nabla_T g_{ij} = 2k_{ij}$ where $k_{ij} = \langle \nabla_T e_i, e_j \rangle$ is the second fundamental form.

$$\begin{aligned}
\nabla_T \mathrm{det} g &= 2k_{ij}G^{ij} \qquad \text{where } G^{ij} \text{ is a cofactor of } g_{ij}.\\
&= 2k_{ij}g^{ij}\mathrm{det} g \qquad \text{where } g^{ij} \text{ is the inverse of g.}\\
&= -2\theta\triangle^2.
\end{aligned}$$

Hence, $\nabla_{\gamma'}\triangle = -\dfrac{1}{2\sqrt{-\mathrm{det} g}}\nabla_{\gamma'}\mathrm{det} g = \theta\triangle.$ □

In conclusion, from the Proposition 4.2.5, q is a conjugate point to Σ along γ if θ tends to $-\infty$ at q. If $q = \gamma(1)$ is a conjugate point, then there exists a sequence of t_n with $t_n < 1$ and $\lim_{n\to\infty} t_n = 1^-$ such that $\lim_{n\to\infty} \theta(t_n) = -\infty$. Also, if $\theta > 0$ at q, we can say the congruence of geodesics starts to diverge at q. If $\theta < 0$ at q, we can say the congruence of geodesics starts to converge at q.

In physics, we usually assume strong energy condition that is $R_{ab}T^aT^b \geq 0$ for all causal vector T. It means gravitation is always an attractive force. With the assumption, we can have some results about the existence of a conjugate point.

Proposition 4.3.2. *Given Σ is either a spacelike hypersurface or a point. Let (γ) be a congruence of timelike geodesics starting from Σ to which γ is orthogonal at $\gamma(0)$. If the expansion θ has a negative value $\theta|_{\gamma_0}(s_1) < 0$ for some point $\gamma_0(s_0) \in (\gamma)$ and if strong energy condition is satisfied everywhere, then there will be a point conjugate to Σ along $\gamma_0(s)$ between $\gamma_0(s_1)$ and $\gamma_0(s_1 + \frac{3}{-\theta})$, provided that $\gamma_0(s)$ can be extended to this parameter value.*

Proof. We show the case for Σ being a spacelike hypersurface only.

Suppose there is no conjugate point between $\gamma(s_1)$ and $\gamma(s_1 + \frac{3}{-\theta})$.

First, we show that $\omega_{ij} = 0$ along γ_0. $\omega_{ij} = \nabla_{[i}T_{j]}$ where $T = \gamma_0'$. It suffices to show that $\nabla_i T_j$ is symmetric. From the first paragraph in the section 4.3 on p.46, there is a map $f : [a,b] \times \Sigma \to M$ around $\gamma_0(0)$ where Σ is locally orthogonal to every timelike geodesic γ. Since $\langle \gamma', \gamma' \rangle = -1$. It is easy to show that $\langle \frac{\partial}{\partial x_i}, \gamma' \rangle = 0$ along γ for $i = 1, 2, 3$. It means, $\frac{\partial}{\partial x_i}(\langle T, \frac{\partial}{\partial x_j} \rangle) = 0$. $\nabla_i T_j = \langle \nabla_{\frac{\partial}{\partial x_i}} T, \frac{\partial}{\partial x_j} \rangle = \frac{\partial}{\partial x_i}(\langle T, \frac{\partial}{\partial x_j} \rangle) - \langle T, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \rangle = \frac{\partial}{\partial x_i}(\langle T, \frac{\partial}{\partial x_j} \rangle) - \langle T, \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} \rangle = \langle \nabla_{\frac{\partial}{\partial x_j}} T, \frac{\partial}{\partial x_i} \rangle = \nabla_j T_i$.

The Raychaudhuri equation along γ_0 becomes $\nabla_T \theta = \frac{1}{3}\theta^2 - \sigma_{ij}\sigma^{ij} - R_{ij}T^iT^j$. Since $\sigma_{ij}\sigma^{ij}$ and $R_{ij}T^iT^j \geq 0$. We have $-\frac{1}{\theta^2}\frac{d}{dt}(\theta) \geq \frac{1}{3}$. So, $\theta^{-1}(t) \geq \theta^{-1}|_{\gamma_0(s_1)} + \frac{t-s_1}{3}$. $\lim_{t \to s_1 - \frac{3}{\theta|_{\gamma_0(s_1)}}} \theta^{-1}(t) \geq 0^-$. From the Raychaudhuri equation, $\theta(t)$ is decreasing and negative. So $\theta(t) \to -\infty$. By the Proposition 4.3.1, there is a contradiction. $\square$

Next, we turn to the behavior of a congruence of null geodesics (γ'). We have the same f mentioned in the first paragraph in the section 4.3 on p.46, but this time $f_{(0,x_1,x_2,x_3)}(t)$ is an integrated null geodesic with initial value $(0, x_1, x_2, x_3) \in \Lambda$. The congruence must have $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle = 0$ and $\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0$. We care about the spacelike space which is normal to a null geodesic. However, $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle = 0$. There are no unique ways to define the two dimensional subspace of spatial vectors normal to $\frac{\partial}{\partial t}$. To solve it, let $\gamma(t) = f(0, x_1, x_2, x_3)(t)$ be a null geodesic passing through p. Then, we choose $N(0)$ to be a null vector at p such that $\langle N, \gamma' \rangle(p) = 1$. Then, we construct a pseudo-orthonormal basis $\{e_1, e_2, N, T\}$ along γ where $T = \gamma'$. We care about the 2 dimension spacelike space spanned by $\{e_1, e_2\}$ which is determined by N and γ' only.

Next, we will derives the Raychaudhuri equation in a null like case.

For each point p, let $\widehat{P}_{ij} = g_{ij} - T_iN_j - N_iT_j$ be the induced metric on the normal subspace spanned by $\{e_1, e_2\}$. It is clear that $\nabla_T P_{ij} = 0$. Also, we consider

$H_{ij} = \widehat{P}_{i\mu}\widehat{P}_{j\nu}\nabla^{\mu}T^{\nu}$ and decompose it into symmetric and antisymmetric part. It is a tensor on the normal subspace. We can use $\widehat{P}^{ij}$ to take trace. We now define expansion $\widehat{\theta}$, shear $\widehat{\sigma_{ij}}$ and rotation $\widehat{\omega_{ij}}$ in a null like case. Expansion $\widehat{\theta} = \widehat{P}^{ij}H_{ij}$. However, we notice that $\widehat{\theta} = \widehat{P}^{ij}H_{ij} = \widehat{P}_{\mu\nu}\nabla^{\mu}T^{\nu} = g_{\mu\nu}\nabla^{\mu}T^{\nu} = \nabla_i T^i = \theta$. The shear $\widehat{\sigma_{ij}} = H_{(ij)} - \frac{1}{2}\theta\widehat{P}_{ij}$. It is symmetric and traceless. The rotation $\widehat{\omega_{ij}} = H_{[ij]}$. It is antisymmetric and traceless.

So, $H_{ij} = \frac{1}{2}\theta\widehat{P}_{ij} + \widehat{\sigma}_{ij} + \widehat{\omega}_{ij}$

When we consider the change of H_{ij} along the null geodesic.

$$\begin{aligned}
\nabla_T H_{ij} &= T^k\nabla_k(\widehat{P}_{i\mu}\widehat{P}_{j\nu}\nabla^{\mu}T^{\nu}) \\
&= \widehat{P}_{i\mu}\widehat{P}_{j\nu}T^k\nabla_k(\nabla^{\mu}T^{\nu}) \\
&= \widehat{P}_i^{\mu}\widehat{P}_j^{\nu}(-\nabla_{\mu}T^k\nabla_k T_{\nu} + R_{k\mu\nu l}T^lT^k) \\
&= -\widehat{P}_i^{\mu}\widehat{P}_j^{\nu}(g_{ij}\nabla_{\mu}T^i\nabla^j T_{\nu}) - \widehat{P}_i^{\mu}\widehat{P}_j^{\nu}R_{\mu k\nu l}T^lT^k \\
&= -\widehat{P}_i^{\mu}\widehat{P}_j^{\nu}(\widehat{P}_{ij}\nabla_{\mu}T^i\nabla^j T_{\nu}) - \widehat{P}_i^{\mu}\widehat{P}_j^{\nu}R_{\mu k\nu l}T^lT^k \\
&= -H_{\mu}^{k}H_{\nu}^{k} - \widehat{P}_i^{\mu}\widehat{P}_j^{\nu}R_{\mu k\nu l}T^lT^k.
\end{aligned}$$

Taking trace of $\nabla_T\nabla_i T_j$, we have

$\nabla_T\theta = -\frac{1}{2}\theta^2 - \widehat{\sigma}_{ij}\widehat{\sigma}^{ij} + \widehat{\omega}_{ij}\widehat{\omega}^{ij} - R_{ij}T^iT^j$. It is the Raychaudhuri equation in the nulllike case.

Let γ be a future null geodesic. We take $\triangle$ to be the volume element mentioned in the Proposition 4.2.6 under $\{e_1, e_2, N, T\}$. By a similar method in the Proposition 4.3.1, we have the followin result.

Proposition 4.3.3. *Let θ be an expansion of a congruence of null geodesics (γ). Then, $\theta = \dfrac{1}{\triangle}\nabla_{\gamma'}\triangle$ along a null like geodesic.*

So,the relation of expansion θ and conjugate point in a null like case is the same as that in the timelike case. Also, referring to the Proposition 4.3.2, we have the following result.

Proposition 4.3.4. *Given Σ is either a spacelike two-surface or a point. Let (γ) be a congruence of null geodesics starting from Σ to which γ is orthogonal at $\gamma(0)$. If the expansion θ has a negative value $\theta|_{\gamma_0}(s_1) < 0$ for some point $\gamma_0(s_1) \in (\gamma)$ and if strong energy condition is satisfied everywhere, then there will be a point conjugate to Σ along $\gamma_0(s)$ between $\gamma_0(s_1)$ and $\gamma_0(s_1 + \frac{2}{-\theta})$, provided that $\gamma_0(s)$ can be extended to this parameter value.*

Apart from the Proposition 4.3.2 and 4.3.4, there is another proposition about the existence of a pair of conjugate points along a causal geodesic.

Definition 4.3.5. *M is said to be satisfied with the generic condition if any causal geodesic contains a point at which $\sum_{c,d=1}^{4} K^c K^d K_{[a} R_{b]cd[e} K_{f]} \neq 0$ where K is the tangent of the geodesic.*

If K is timelike, then we can have an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ mentioned in the Definition 4.1.4 where $e_4 = K$. We have that $\sum_{c,d=1}^{4} K^c K^d K_{[a} R_{b]cd[e} K_{f]} \neq 0$ at a point implies $R_{b44e} \neq 0$ for some $1 \leq b, e \leq 3$ at the point. If K is null, then we also have a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ mentioned in the Definition 4.2.3 where $e_3 = N$ and $e_4 = K$. We have that $\sum_{c,d=1}^{4} K^c K^d K_{[a} R_{b]cd[e} K_{f]} \neq 0$ at a point implies $R_{b44e} \neq 0$ for some $1 \leq b, e \leq 2$ at the point.

Lemma 4.3.6. *For any $s^* > 0$, there exists $c > 0$ such that if $max|a'_{ij}(0)| \geq c$, $a(0) = I_{3\times 3}$, $tr(a'(0)) \leq 0$, $a'(0)$ is symmetric, $a''_{ij} + a_{ik} R_{k44j} = 0$, then $det(a(s_1)) = 0$ for some $s_1 \in [0, s^*]$.*

Proof. As $a''_{ij} + a_{ik} R_{k44j} = 0$ is a linear ODE. It suffices to show for any $s^* > 0$, there exists $\epsilon > 0$ such that if $\max|a'_{ij}(0)| = 1$, $a(0) = \epsilon_a I$ where $\epsilon_a < \epsilon$, $\mathrm{tr}(a'(0)) \leq 0$, $a'(0)$ is symmetric, $a''_{ij} + a_{ik} R_{k44j} = 0$, then $\det(a(s_1)) = 0$ for some $s_1 \in [0, s^*]$.

By the Taylor's expansion, we have $a_{ij}(s) = \epsilon_a I + s a'_{ij}(0) + \frac{s^2}{2} a''_{ij}(\xi_{ij})$ where $0 \le \xi_{ij} \le s$.

For $a''_{ij}(\xi_{ij})$, $a(0)$ is bounded by ϵI while $a'_{ij}(0)$ is bounded by 1. Thus, from $a''_{ij} + a_{ik} R_{k44j} = 0$, we have $a(s)$ on $[0, s^*]$ is bounded by a constant which is independent choice of a. Also, R_{k44j} is bounded on $[0, s^*]$. Hence, a''_{ij} is bounded on $[0, s^*]$. Hence, $|a''_{ij}(\xi_{ij})| \le C_{s^*}$ where C_{s^*} is independent for the choice of $a(0)$ and $a'(0)$.

Next, $a'_{ij}(0)$ is symmetric. Let λ_1, λ_2 and λ_3 be eigenvalues of $a'_{ij}(0)$ with $\lambda_1 \le \lambda_2 \le \lambda_3$. We claim $\lambda_1 \le -\frac{1}{2}$. We have $\lambda_1 = \min_{\{x_1^2+x_2^2+x_3^2=1\}} a'_{ij}(0) x_i x_j$. There are two cases to consider.

Case 1. $|a'_{kk}(0)| = 1$ for some k. W.L.O.G., we assume $k = 1$. For $a'_{11}(0) = 1$, $a'_{11}(0) + a'_{22}(0) + a'_{33}(0) \le 0 \Rightarrow a'_{22}(0) + a'_{33}(0) \le -1$. It means either $a'_{22}(0)$ or $a'_{33}(0)$ less than or equal to $-\frac{1}{2}$. By taking suitable x, we have $\lambda_1 \le a'_{kk}(0)$ where k = 1,2 or 3. As a result, we have $\lambda \le -\frac{1}{2}$. Also, for $a'_{11}(0) = -1$, we have $\lambda_1 \le a'_{11}(0) = -1$.

Case 2. $|a'_{ij}(0)| = 1$ for some $i \ne j$. W.L.O.G, we assume $i = 1$ and $j = 2$. For $a_{12} = 1$, we let $x = (1, -1, 0)$. We have $\lambda_1 \le a'_{11}(0) - 2a'_{12}(0) + a'_{22}(0) = a'_{11}(0) - 2 + a'_{22}(0)$. If $a'_{11}(0) + a'_{22}(0) \le 1$, then $\lambda_1 \le -1$. If $a'_{11}(0) + a'_{22}(0) \ge 1$, then $a'_{33}(0) \le -1$. λ_1 is still less than or equal to -1. For $a_{12} = -1$, we take $x = (1, 1, 0)$ and repeat the above process. We also find that $\lambda_1 \le -1$.

By the above, $|a'_{ij}(0) + \frac{s}{2} a''_{ij}(\xi_{ij}) - a'_{ij}(0)| \le \frac{s}{2} C_{s^*}$ and $|a'_{ij}(0)| \le 1$. Also, eigenvalues of a 3 by 3 matrix are roots of a cubic equation which has an explicit formula to solve. There exists $s_0 \in [0, s^*]$ which is independent of $a(0)$ and $a'(0)$ such that $a'_{ij}(0) + \frac{s_0}{2} a''_{ij}(\xi_{ij})$ has an eigenvalue $\le -\frac{1}{4}$. So $a'_{ij}(s_0) = \epsilon_a I + s_0(a'_{ij}(0) + \frac{s_0}{2} a''_{ij}(\xi_{ij}))$. By the similar argument about the existence of s_0, we can make ϵ small enough such that $\epsilon_a I + s_0(a'_{ij}(0) + \frac{s_0}{2} a''_{ij}(\xi_{ij}))$ has an eigenvalue $\le -\frac{s_0}{8}$ for all $0 < \epsilon_a < \epsilon$. All eigenvalues of $a(0)$ is $\epsilon_a > 0$, but there is a negative eigenvalue of $a_{ij}(s_0)$. By the mean value theorem, there is a zero eigenvalue of $a_{ij}(s_1)$ for some $s_1 \in [0, s_0]$. So $\det(a)(s_1) = 0$. $\square$

Proposition 4.3.7. *If the strong energy holds and generic condition is satisfied, then there is a pair of conjugate points along any causal geodesic provided that the geodesic can extend up to the conjugate point.*

Proof. Let γ be a timelike geodesic. Since the space-time satisfy the generic condition, W.L.O.G., we can assume $\sum_{c,d=1}^{4} K^c K^d K_{[a} R_{b]cd[e} K_{f]} \neq 0$ at $\gamma(0)$ where $K = \gamma'(0)$. We consider a congruence of timelike geodesics containing γ. Let $\{e_1(t), e_2(t), e_3(t), \gamma'(t)\}$ be an orthonormal basis along γ. By the Theorem 4.2.2, we have $\nabla_{\gamma'}\nabla_{\gamma'} J_i(s) + R(J_i, \gamma')\gamma' = 0$ where $J_i(0) = e_i$ and $J_i'(0) = \nabla_{e_i}\gamma'$. Since $J_i(s)$ is orthogonal to $\gamma(s)$, we have $J_i(s) = \sum_{j=1}^{3} a_{ij}(s) e_j(s)$. It means $a_{ij}'' + a_{ik} R_{k44j} = 0$ along γ.

We let $S = \{b$ is a 3 by 3 symmetric matrix with $\mathrm{tr}(b) \leq 0\}$. We claim for any $b \in S$, if $a_{ij}(0) = I_{3\times 3}$, $a_{ij}'(0) = b$ and $a_{ij}'' + a_{ik} R_{k44j} = 0$, then $\det(a(s)) = 0$ for some $s > 0$. On p.83 in [7], $\omega_{ij}(s) = -a^{-1}_{k[i} a'_{j]l}(s)$. So $\omega_{ij}(0) = 0$. Also, on p.83 in [7], we have $\frac{d}{ds}(a_{ij}\omega_{ik}a_{kl}) = 0$. It means $\omega = 0$ along γ. For $\mathrm{tr}(b) < 0$, then by the Proposition 4.3.1, we have $\theta = \mathrm{tr}(a_{ij}' A_{ij})$ where A_{ij} is an inverse of a_{ij}. Hence, $\theta(0) < 0$. By the Proposition 4.3.2, $\det(a(s)) = 0$ for some $s > 0$. For $\mathrm{tr}(b) = 0$, by the Proposition 4.3.1, we have $\theta = 0$ at $\gamma(0)$. Suppose $R_{ab}\gamma'^a\gamma'^b + \sigma_{ab}\sigma^{ab} = 0$ at $\gamma(0)$. It implies $\sigma_{ab} = 0$. On p.218 in [13], we have $\frac{d}{ds}\sigma_{ab} = C_{4ba4} + \frac{1}{2}\widetilde{R}_{ab} = -R_{a4b4}$ at $\gamma(0)$. Since $\sum_{c,d=1}^{4} K^c K^d K_{[a} R_{b]cd[e} K_{f]} \neq 0$ at p where $K = \gamma'(0)$, $\frac{d}{ds}\sigma_{ab} \neq 0$ for some a and b at $\gamma(0)$. $\sigma_{ab}\sigma^{ab} > 0$ locally around $s > 0$. By Raychaudhuri equation, $\frac{d}{ds}(\theta) < 0$. It means $\theta < 0$ around $s > 0$. Again, by the Proposition 4.3.2, we have $\det(a(s)) = 0$ for some $s > 0$. Suppose $R_{ab}\gamma'^a\gamma'^b + \sigma_{ab}\sigma^{ab} > 0$, then by Raychaudhuri equation, we also have $\det(a(s)) = 0$ for some $s > 0$.

Let $\eta : S \to [0, +\infty)$ such that $\eta(b) = \min\{s \in [0, +\infty) | \det(a(s)) = 0$ with $a_{ij}(0) = I_{3\times 3}, a_{ij}'(0) = b$ and $a_{ij}'' + a_{ik} R_{k44j} = 0\}$. We claim η is continuous.

Suppose the map is not continuous at some $b \in S$. There exists a $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there exists some $b_n \in S$ with $\max|(b_n)_{ij} - b_{ij}| < \frac{1}{n}$ but $|\eta(b_n) - \eta(b)| > \epsilon$. There are two cases to consider.

Case (1.) $\mathrm{tr}(b) < 0$ at $\gamma(0)$. There exists $N \in \mathbb{N}$ such that $\mathrm{tr}(b_n) < \dfrac{\mathrm{tr}(b)}{2}$ for $n > N$. By the Proposition 4.2.5, 4.3.1 and 4.3.2, $\eta(b_n) \in [0, \frac{-6}{tr(b)}]$. W.L.O.G., we can assume $\eta(b_n)$ converges to ξ for some $\xi \geq 0$. By smoothness of ODE, we find that $\det(b(\xi)) = 0$. If $\xi < \eta(b)$, there is a contradiction since $\eta(b)$ is the first point of $\det(b) = 0$. If $\xi > \eta(b)$, then there is a sequence of expansion $\theta(h)$ with respect to b such that $\theta(h)$ tends to $-\infty$ when h tends to $\eta(b)^-$. So by the smoothness of ODE and the Proposition 4.3.2, for large n, the expansion of b_n is so small that its first point of $\det(b_n) = 0$ lies inside $(\eta(b) - \epsilon, \eta(b) + \epsilon)$. Contradiction again.

Case (2.) $\mathrm{tr}(b) = 0$ at $\gamma(0)$. By the above argument, we have $\mathrm{tr}(b) < 0$ around p. By the argument in the case (1.), we have a contradiction again.

Next, we claim there exists $s_1 > 0$ such that $\eta(S) \subseteq [0, s_1]$. We let $c > 0$ be a constant. Let $S_c = \{b \in S | \max|b_{ij}| > c\}$. By the Lemma 4.3.6, $\eta(S_c)$ is bounded. $S - S_c$ is a compact set. η is continuous so $\eta(S - S_c)$ is bounded. The claim is done.

Finally, we take $s_2 > s_1$. Suppose there is no conjugate to $\gamma(s_2)$ along $\gamma[0, s_2]$. Otherwise, the proposition is done. There exists a Jacobi field $T_i(s)$ along γ with $T_i(0) = e_i$, $T_i(s_2) = 0$. Let $A(s) = [T_1(s), T_2(s), T_3(s)]$ under $\{e_1, e_2, e_3\gamma'\}$. On p.97 in [7], $A_{ki}\omega_{kl}A_{lj} = \frac{1}{2}(A_{ki}\frac{d}{ds}A_{kj} - A_{kj}\frac{d}{dt}A_{ki})$ will be constant along $\gamma[0, s_2]$. So, $A_{ki}\omega_{kl}A_{lj} = 0$ at $s = s_2$. Also, A_{ij} has an inverse on $\gamma[0, s_2)$. Therefore, $\omega_{ij} = 0$ on $\gamma[0, s_2)$ and $A'_{ij}(0)$ is symmetric. Since $s_2 \notin [0, s_1]$, we have $A(0) \notin S$ which means $\mathrm{tr}(A'(0)) > 0$. By the Proposition 4.2.5 and 4.3.2, there exists $s_3 < 0$ such that $\det(A)(s_3) = 0$. It means there exists constant c_1,c_2 and c_3 with some $c_i \neq 0$ such that $\sum_{i=1}^{3} c_i T_i(s_3) = 0$. Also, it is clear that $\sum_{i=1}^{3} c_i T_i(s_2) = 0$. There exists a non-trivial Jacobi field $\sum_{i=1}^{3} c_i T_i$ which vanishes at s_2 and s_3. It means $\gamma(s_2)$

and $\gamma(s_3)$ is a pair of conjugate points.

Let γ be a null geodesic. Since the space-time satisfy the generic condition, W.L.O.G., we can assume $\sum_{c,d=1}^{4} K^c K^d K_{[a} R_{b]cd[e} K_{f]} \neq 0$ at $\gamma(0)$ where $K = \gamma'(0)$. We only modify slightly the above argument for a timelike geodesic to reach the conclusion. $\square$

There is a remark about the Proposition 4.3.2, 4.3.4 and 4.3.7. The proposition may fail if a causal geodesic is incomplete.

Chapter 5

Singularity Theorems

In section 5.1, we will define what is a singularity in space-time. We will prove two singularity theorems. In section 5.2, we will study the singularity theorem in [8]. We will show that strong energy condition for null vectors, the existence of a trapped surface and a non-compact Cauchy surface implies the existence of singularities. In section 5.3, we will study another singularity theorem in [6]. We will show that strong energy condition, generic condition, chronology condition and the existence of a trapped set implies the existence of singularities.

5.1 Definition of Singularities in Space-Time

When we study singularities in a space-time manifold. The manifold must be inextendible since we do not want to say , for example, $(R^4 - \{(0,0,0,0)\}, -dt^2 + dx^2 + dy^2 + dz^2)$ has a singular point at the origin since it can be simply removed by the isometric extension of the space. In order to define a singularity, we first define inextendible space-time.

Definition 5.1.1. *Let (M, g) be a space-time manifold. If for any space-time $(\widetilde{M}, \widetilde{g})$ and a one to one and C^1 map $f : M \to \widetilde{M}$ with $\widetilde{g}|_{f(M)} = f_* g$ and $f|_{f(M)}$*

diffeomorphic to M, then we have $f(M) = \widetilde{M}$. (M, g) is said to be inextendible

Next, we define a generalized affine parameter μ.

Definition 5.1.2. *For any C^1 curve $\gamma : [a, b] \to M$. Let $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ be a basis of $T_{\gamma(a)}M$. We parallel transport ω_i along γ. Then, $\gamma'(t) = \sum_{i=1}^{4} v_i(t)\omega_i(t)$. A generalized affine parameter $\mu(t) = \int_a^t \sqrt{\sum_{i=1}^{4} v_i^2(s)}ds$. Also, γ is said to have a finite arc-length in μ if and only if $\mu(t)$ is finite for $t \in [a, b]$.*

Remark: It is necessary for γ to be C^1 because

(1.) Parallel transport along γ to be γ is C^1;

(2.) $\sqrt{\sum_{i=1}^{4} v_i^2(s)}$ is required to be integrable on $[a, b]$.

Proposition 5.1.3. *γ has a finite arc-length in the generalized affine parameter μ if and only if γ has finite arc-length in any other generalized affine parameter λ.*

Proof. For any two basis of $T_{\gamma(t)}M$, $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ which are parallel transported along γ, we have $\gamma' = \sum_{i=1}^{4} v_i\omega_i = \sum_{i=1}^{4} u_i\eta_i$. W.L.O.G., we let $\mu(t) = \int_a^t \sqrt{\sum_{i=1}^{4} v_i^2(s)}ds$ and $\lambda(t) = \int_a^t \sqrt{\sum_{i=1}^{4} u_i^2(s)}ds$. There exists a constant and non-degenerate 4×4 matrix a and its inverse A such that $u_i = \sum_{j=1}^{4} a_{ij}v_j$ and $v_i = \sum_{j=1}^{4} A_{ij}u_j$. We have $|u_i| \leq \sum_{j=1}^{4} |a_{ij}||v_j| \leq \max_{ij} |a_{ij}| \sum_{j=1}^{4} |v_j|$. Hence, we have $\sum_{i=1}^{4} |u_i|^2 \leq 4 \max_{ij} |a_{ij}|^2 (\sum_{j=1}^{4} |v_j|)^2 \leq 16 \max_{ij} |a_{ij}|^2 \sum_{j=1}^{4} |v_j|^2$.

Similarly, $\sum_{i=1}^{4} |v_i|^2 \leq 16 \max_{ij} |A_{ij}|^2 \sum_{j=1}^{4} |u_j|^2$. So, there exist c_1 and c_2 such that

$c_1^2 \sum_{j=1}^{4} |u_j|^2 \leq |v_i|^2 \leq c_2^2 \sum_{j=1}^{4} |u_j|^2$. As a result, $c_1\mu(t) \leq \lambda(t) \leq c_2\mu(t)$. □

Corollary 5.1.4. *A causal geodesic has a finite length under affine parameter if and only if it has a finite length under generalized affine parameter.*

Proof. For a timelike geodesic $\gamma : (0, a) \to M$ which is parametrized by an affine parameter. Let $\{e_1, e_2, e_3, \gamma(s)\}$ be an orthonormal basis along γ. Then, $\mu(t) = \int_0^t ds = t$ is a generalized parameter where $0 \leq t \leq a$. It means affine parameter is generalized affine paramter. By the Proposition 5.1.3, the corollary is done. Similarly, for a null geodesic γ, the corollary is also true. □

With the Proposition 5.1.3, we can define b-complete.

Definition 5.1.5. (M, g) *is b-complete if and only if there is an endpoint for every* C^1 *curve of finite length as measured by a generalized affine parameter.* (M, g) *is b-incomplete if* (M, g) *is not b-complete.*

A causal geodesic is said to be complete if its maximum domain under affine parameter is the whole $\mathbb{R}$, otherwise it is incomplete. Hence, by the Corollary 5.1.4, a causal geodesic is incomplete if and only if it is b-incomplete.

Finally, we can talk about singularities in space time with the concept of an inextendible space-time and b-complete.

Definition 5.1.6. *A space-time manifold is singularity-free if it is inextendible and b-complete. A space-time manifold is singular if it is not singularity-free.*

If you are interested in the motivation of the definition of singularities in space-time, you can read from p.256 to p.261 in [7].

5.2 A Singularity Theorem of R. Penrose

In 1965, R. Penrose found that there is a close relationship between a trapped surface and singularities in globally hyperbolic space-time. First, we define a trapped surface.

Definition 5.2.1. Γ *is a* C^2 *closed (compact without boundary) spacelike two-surface such that there exist two smooth congruence of future linear independent null geodesic,* (γ_1) *and* (γ_2), *passing every point on* Γ *orthogonally with* $\theta_{\gamma_i} < 0$ *on* Γ *for* $i = 1$ *and* 2. *Then,* Γ *is said to be future trapped surface. Similarly, we can define a past trapped surface.*

We will prove the singularity theorem 1.

Theorem 5.2.2. *[8] Space-time* (M, g) *cannot be null geodesic complete if*

(1.) $R_{ab}K^aK^b \geq 0$ *for all null vector* K^a*;*

(2.) there is a non-compact Cauchy Surface K *in* M*;*

(3.) there is a trapped surface Γ *in* M*.*

Proof. We divide the proof into two parts.

(I.) Under the conditions (1.) and (3.), $\partial J^+(\Gamma)$ is compact if M were null geodesically complete.

(II.) Compact $\partial J^+(\Gamma)$ is incompatible with the condition (2.)

First, we will prove part I.

Suppose M is null geodesically complete.

We will show that $J^+(\Gamma)$ is closed. By the Theorem 3.3.9 and the condition (2.), M is globally hyperbolic. Then, we claim that $J^+(p)$ is closed for any

$p \in \Gamma$. Suppose $q \in \overline{J^+}(p) - J^+(p)$. Since $I^+(q)$ is open, there exists $r \neq q$ such that $r \in I^+(q)$. Then, $q \in \overline{J^+}(p) \cap I^-(r) \subseteq \overline{J^+(p) \cap J^-(r)} = J^+(p) \cap J^-(r)$. Contradiction. The claim is done. Finally, we will show that $J^+(\Gamma)$ is closed. Suppose $q \in \overline{J^+(\Gamma)} - J^+(\Gamma)$. Let $q_n \in I^+(\Gamma)$ with $q_{n+1} \ll q_n$ and $q_n \to q$. By the above, $J^-(q_n) \cap \Gamma$ is a non-empty compact nested sequence, so $\cap_{n=1}^{\infty} J^-(q_n) \cap \Gamma$ is non-empty. Let say $p \in \cap_{n=1}^{\infty} J^-(q_n) \cap \Gamma$. Then $p \ll q_n$ for all n. It means $q \in \overline{J^+(p)} = J^+(p) \subseteq J^+(\Gamma)$. Contradiction.

Then, we will show $\partial J^+(\Gamma)$ is non-empty and generated by null geodesics which have endpoint on Γ and are orthogonal to it.

To show $\partial J^+(\Gamma)$ is non-empty, we first claim $J^+(\Gamma)$ cannot be open. Suppose $J^+(\Gamma)$ is open. Since $J^+(\Gamma)$ is closed, we have $J^+(\Gamma) = M$. Hence, Γ is covered with $\{I^+(p) | p \in \Gamma\}$. Since Γ is compact, it can be covered with $I^+(p_1)$,...,$I^+(p_n)$ for some n. Then, there is a closed timelike curve in M. It contradicts that M is globally hyperbolic. The claim is done. Hence, $\partial J^+(\Gamma) = J^+(\Gamma) - I^+(\Gamma)$ is non-empty since $I^+(\Gamma)$ is open.

For any $p \in \partial J^+(\Gamma)$, there is a past causal curve $\gamma : [0,1] \to M$ from $p = \gamma(0)$ to $q = \gamma(1)$ for some $q \in \Gamma$. Then, γ must be null geodesic, otherwise $p \in I^+(\Gamma)$.

Also, suppose γ is not orthogonal to Γ at q. W.L.O.G., $\gamma[0,1]$ lies inside a convex normal neighbourhood U of $\gamma(1)$. Then, there exists $\alpha(u)$ be a smooth curve on Γ passing through $\gamma(1)$ at $u = 0$ such that $\langle \alpha'(0), \gamma'(1) \rangle \neq 0$ and $\alpha|_{(-\epsilon,\epsilon)} \subseteq \Gamma$ for some small $\epsilon > 0$. Let $\beta(u,t) : (-\epsilon, \epsilon) \times [0,1] \to M$ be a variation of geodesic such that $\beta(u,[0,1])$ is the geodesic from $\gamma(0)$ to $\alpha(u)$ inside U. We let $L(\beta(u,[0,1])) = \int_0^1 -\langle \frac{\partial \beta}{\partial t}, \frac{\partial \beta}{\partial t} \rangle dt$. Then the first variational formula is $\frac{1}{2}\frac{\partial L}{\partial u}|_{u=0} = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial t} \rangle|_\Sigma + \int_0^1 \langle \frac{\partial}{\partial u}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} \rangle dt = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial t} \rangle|_\Sigma \neq 0$. Therefore, $L(\beta(0,[0,1]))$ is not a critical point. There exists a sequence u_n with $\lim_{n\to\infty} u_n = 0$ such that $L(\beta(u_n,[0,1]) < 0$ for all n. Hence, $\beta(u_n,t)$ is a timelike geodesic. There are two cases to consider.

Case (1.) $\beta(u_n, [0,1])$ is a future timelike geodesic for all n. Then, we have $\gamma(1) \preceq \gamma(0) \ll \beta(u_m, 1)$. As $\beta(u_n, 1)$ tends to $\gamma(1)$ and $\beta(u_n, [0,1]) \subseteq U$. By the smoothness of ODE, we have $\gamma(0) \preceq \gamma(1)$. As a result, we have a closed causal curve which contradicts globally hyperbolic M.

Case (2.) There exists $\beta(u_n, t)$ is a past timelike curve. Then, $\gamma(0) \in I^+(\Gamma)$ which contradicts with $\gamma(0) \in \partial J^+(\Gamma)$.

Finally, we can show $\partial J^+(\Gamma)$ is compact to complete the proof of the part $(I.)$. There are two smooth congruence of future linear independent null geodesics (γ_1) and (γ_2) which start from every point on Γ orthogonally, $\gamma_i(0) \in \Gamma$ and $\theta_{\gamma_i}(0) < 0$. Since M is null geodesic complete, by the Proposition 4.3.4, the first conjugate point to Γ along γ_i lies in $(0, \dfrac{2}{\theta_{\gamma_i}(0)}]$. Since $\theta_{\gamma_i}(0)$ is continuous on Γ and Γ is compact. There exists $b > 0$ such that for any null geodesic γ_i, its first conjugate point to Γ lies in the open interval $(0, b)$. Then we let a map

$$\beta : \Gamma \times [0, b] \times \{1, 2\} \to M$$

such that $\beta(p, t, i)$ maps to $\gamma_i(t)$ with $\gamma_i(0) = p$. By the Theorem 4.2.11 and the second paragraph of the part (I.), we have $\partial J^+(\Gamma) \subseteq \beta(\Gamma \times [0, b] \times \{1, 2\})$. By the smoothness of ODE theorem, β is continuous. $\Gamma \times [0, b] \times \{1, 2\}$ is compact and $\partial J^+(\Gamma)$ is closed. As a result, $\partial J^+(\Gamma)$ is compact.

Next, we will prove the part $(II.)$.

There is a smooth timelike vector field on M because M is time-orientable. We assume those integrated future timelike curves λ meets Γ at $\lambda(0)$. There is a map $T : \partial J^+(\Gamma) \to K$ such that $\lambda(0)$ maps to $\lambda(-\infty, \infty) \cap K$. The map is well-defined since K is Cauchy surface.

We will claim $\partial J^+(\Gamma)$ is homeomorphic to $T(\partial J^+(\Gamma))$.

First, suppose T is not injective. There exists $p \neq q$ with $T(p) = T(q)$. There exists the integrated future timelike curves λ such that passing through p, $T(p) = T(q)$ and q. It shows p and q has a timelike relation. It contradicts that $\partial J^+(\Gamma)$

is achronal.

Then, we will show T is continuous. Let d be the natural distance function between p and $q \in M$ with respect to a Riemannian metric on M. For any $q \in \partial J^+(\Gamma)$, any sequence $\{q_n\} \subseteq \partial J^+(\Gamma)$ which converges to q, we let $T(q) = \lambda_q(t_q)$ where $\lambda_q(0) = q$ and $T(q_n) = \lambda_{q_n}(t_{q_n})$ where $\lambda_{q_n} = q_n$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $d(\lambda_q(t), \lambda_q(t_q)) < \epsilon$ for $t \in [t_q - \delta, t_q + \delta]$. By smoothness of ODE theorem, there exists N such that $d(\lambda_{q_n}(t), \lambda_q(t)) < \epsilon$ for $t \in [t_q - \delta, t_q + \delta]$ and $n \geq N$. Since Also, $\lambda_q(t_q - \delta)$ lies in $I^-(K)$ while $\lambda_q(t_q + \delta)$ lies in $I^+(K)$. We can make N larger such that $\lambda_{q_n}(t_q - \delta) \in I^-(K)$ and $\lambda_{q_n}(t_q + \delta) \in I^+(K)$ for all $n \geq N$. Since K is Cauchy surface, we have $\lambda_{q_n}(t_{q_n}) \in (t_q - \delta, t_q + \delta)$ for $n \geq N$. As a result, for $n \geq N$,

$$
\begin{aligned}
d(T(q_n), T(q)) &= d(\lambda_{q_n}(t_{q_n}), \lambda_q(t_q)) \\
&\leq d(\lambda_{q_n}(t_{q_n}, \lambda_q(t_{q_n})) + d(\lambda_q(t_{q_n}, \lambda_q(t_q)) \\
&\leq 2\epsilon.
\end{aligned}
$$

By the part $(I.)$, $\partial J^+(\Gamma)$ is compact, $T(\partial J^+(\Gamma))$ is Hausdorff space and $T : \partial J^+(\Gamma) \to T(\partial J^+(\Gamma))$ is a bijective continuous function. The claim is done.

Finally, we show $T(\partial J^+(\Gamma)) = K$. It is clear that $T(\partial J^+(\Gamma)) \subseteq K$. By the Proposition 3.3.6, it suffices to show $T(\partial J^+(\Gamma))$ is non-empty, open and closed in K. It is easy to show that $T(\partial J^+(\Gamma))$ is non-empty and closed. To show it is open, by the Lemma 3.1.2, for any $p \in T(\partial J^+(\Gamma))$, there exists a coordinate map ϕ_1, open U containing p such that $\phi_1 : U \cap T(\partial J^+(\Gamma)) \to \phi_1(U \cap T(\partial J^+(\Gamma))) \subseteq R^3$ is a homeomorphism and $\phi_1(U \cap T(\partial J^+(\Gamma)))$ is open in R^3. Also, K is a Cauchy surface means $K = \partial J^+(K)$. If U is small, there exists a coordinate map ϕ_2 such that $\phi_2 : U \cap K \to \phi_2(U \cap K) \subseteq R^3$ is a homeomorphism and $\phi_2(U \cap K)$ is open in R^3. Then $\phi_2 \circ \phi_1^{-1}$ is injective continuous. By invariance of domain, $\phi_2 \circ \phi_1^{-1}(U \cap T(\partial J^+(\Gamma)))$ is open is R^3. Hence, $\phi_2 \circ \phi_1^{-1}(U \cap T(\partial J^+(\Gamma)))$ is open in $\phi(U \cap K)$. It means $p \in U \cap T(\partial J^+(\Gamma)) \subseteq \partial J^+(\Gamma)$ is open in K. Thus, $\partial J^+(\Gamma)$

is open in K.

The part $(II.)$ is done since K is non-compact but $T(\partial J^+(\Gamma))$ is compact. Contradiction. $\square$

5.3 A Singularity Theorem of S.W. Hawking and R. Penrose

Although the singularity theorem in [8] successfully show some physical conditions for the existence of singularities, [6] states that the assumption of a global Cauchy surface is not a good condition on p.530. It is because its existence is hard to justify from the standpoint of general relativity. Also, it is violated in a number of exact models. Thus, S.W. Hawking and R. Penrose published the second singularity theorem which do not require the existence of a global Cauchy surface in 1970 in [6]. We will prove it here. In their paper, their theorem is based on a technical lemma as below.

Definition 5.3.1. *A future-trapped set is a non-empty achronal closed set $S \subseteq M$ for which $E^+(S) = J^+(S) - I^+(S)$ is a compact set. Similarly, we can define a past-trapped set, too.*

Lemma 5.3.2. *No space-time M can satisfy all of the following three requirements together.*

(A) M contains no closed timelike curves;

(B) Every inextendible casual geodesic in M contains a pair of conjugate points;

(C) There exists a future trapped set $S \subseteq M$.

Proof. Suppose the lemma is false. The space-time M do exists. We will prove the following five main parts to draw a contradiction.

(I.) Use (A) and (B) to show M is strongly causal.

(II.) $H^+(E^+(S))$ is non-compact or empty.

(III.) There exists a future-inextendible timelike curve $\gamma : [0, \infty) \to M$ contained in $\mathrm{int}(D^+(E^+(S))$.

(IV.) There exists a past inextendible timelike curve $\lambda : (-\infty, 0] \to M$ contained in $\mathrm{int}(D^-(E^-(F))$ where $F = E^+(S) \cap J^-(\gamma)$

(V.) With γ and λ, we can construct an inextendible causal geodesic μ in $D(E^-(F))$ which is incompatible with the condition (B).

We will prove part (I.).

Suppose strong causality fails at $p \in M$. Let N be a convex normal neighbourhood of p. Following the first paragraph of the Lemma 3.2.5, we have the corresponding Q_i, a_i, b_i,c_i and d_i. Since a_i, d_i converges to p and c_i can be assumed to converge to some point c on ∂N. $a_i \ll c_i \ll d_i$ implies $p \preceq c \preceq p$. $\widehat{pc}$ is a future causal geodesic lying in N from p to c while $\widehat{cp}$ is a future causal geodesic lying in N from c to p. The condition (A) implies $\eta = \widehat{pc} \cup \widehat{cp}$ must be a single null geodesic lying in N. Also, η must be inextendible. By the condition (B), it must have a pair of conjugate points along η. By the Theorem 4.2.12, we can still construct a closed timelike curve which contradicts with the condition (A). So, the part (I.) is true.

We will prove the part (II.).

First, we will claim $H^+(E^+(S)) \subseteq H^+(\partial J^+(S))$. Suppose $x \in H^+(E^+(S)) - H^+(\partial J^+(S))$. Since $E^+(S)$ is closed and achronal, by the item 4 of the Proposition 3.3.5, we have $x \in \overline{D^+(E^+(S))}$. Since $E^+(S) \subseteq \partial J^+(S)$ and by the item 4 of the Proposition 3.3.5, we have $x \in D^+(\partial J^+(S)) - H^+(\partial J^+(S))$. By the definition of $H^+(\partial J^+(S))$, there exits $y \in I^+(x) \cap D^+(\partial J^+(S))$. Also, $x \in H^+(E^+(S))$ means $y \notin D^+(E^+(S))$. There exists a past endless causal curve γ from y which does not cut $E^+(S)$. However, $y \in D^+(\partial J^+(S))$ means γ cuts a point z at $\partial J^+(S)$. By the Corollary 3.1.4, there exists a past null geodesic segment on $\partial J^+(S)$ from z which is either past-endless on $\partial J^+(S)$ or has a past end-point on edge(S).

For the case of the existence of the past end-point on edge(S), closed S means the end-point is in S. Then, $z \in \partial J^+(S) \cap J^+(S) = E^+(S)$. It contradicts with $\gamma \cap E^+(S) = \emptyset$.

For the case of the endless null geodesic on $\partial J^+(S)$, we let the geodesic be η. Since $y \in D^+(\partial J^+(S)) - H^+(\partial J^+(S))$, by the Lemma 3.3.7, every past endless causal curve from y must intersect $I^-(\partial J^+(S))$. Hence, $\eta \cap I^-(\partial J^+(S))$ is non-empty. It contradicts with achronal property of $\partial J^+(S)$.

The claim is done.

Suppose $H^+(E^+(S))$ is non-empty and compact. By the part (I.), M is strongly causal. $H^+(E^+(S))$ is covered with a finite number of causally convex neighbourhood $U_1, U_2, ..., U_n$ which have a compact closure. W.L.O.G, we can assume $z_1 \in U_1 \cap H^+(E^+(S))$. By the claim, we have $z_1 \in U_1 \cap H^+(\partial J^+(S))$. By the item 2 and 3 of the Proposition 3.3.5 and the definition of $H^+(\partial J^+(S))$, there exists x_1 lying in $[U_1 - D^+(\partial J^+(S))] \cap I^+(z_1) \cap I^+(S)$. By the Proposition 3.3.4, there exists a past endless timelike curve α_1 from x_1 such that $\alpha_1 \cap \partial J^+(S) = \emptyset$.

Since U_1 is causally convex set and $\overline{U_1}$ is compact, there must exist $T_1 > 0$ such that $\alpha_1(t) \notin U_1$ for $t \geq T_1$. Otherwise, α_1 is not endless. Also, since $\alpha_1 \cap \partial J^+(S) = \emptyset$ and $\alpha_1(0) = x_1 \in I^+(S)$, we have $\alpha \subseteq I^+(S)$. Thus, there exists a past timelike curve $\beta_1 : [0, 1] \to M$ such that $\beta_1(0) = \alpha_1(T_1)$ and $\beta_1(1) \in S$. On the other hand, $S \subseteq D^+(E^+(S))$ and $\alpha_1 \cap D^+(E^+(S)) = \emptyset$ since $\alpha_1 \cap \partial J^+(S) = \emptyset$. It is easy to conclude that there exists $\xi \in [0, 1]$ such that $\beta_1(\xi) \in H^+(E^+(S))$. Hence, we let the past timelike curve from x_1 as

$$\alpha(t) = \begin{cases} \alpha_1(t) & \text{if } t \in [0, T_1] \\ \beta_1(t - T_1) & \text{if } t \in (T_1, T_1 + \xi] \end{cases}$$

It will not meet U_1 when $t \geq T_1$ since U_1 is strongly causal and $x_1 \in U_1$. In particular, we have $\beta_1(\xi) \in H^+(E^+(S)) - U_1$. Then , we let $z_2 = \beta_1(\xi)$ and W.L.O.G., we assume z_2 lies in U_2. By the above method, we can extend α which don't meet U_1 and U_2 after $\alpha(T_2)$ for some T_2. After we repeat the above process more than n times, the past timelike curve α must intersect U_i more than once for some $i = 1, ..., n$. Contradiction. The part (II.) is done.

We will show the part (III.).

Suppose all future inextendible timelike curves $\gamma \nsubseteq D^+(E^+(S))$. There exists a smooth timelike vector field on M. We have a smooth family of future endless timelike integrated curves α on M. We let $T : E^+(S) \to H^+(E^+(S))$ such that $p \in E^+(S)$ maps to $\alpha \cap H^+(E^+(S))$ where α is the integrated curve passing through p. Since $H^+(E^+(S))$ is achronal and $\alpha \nsubseteq D^+(E^+(S))$, every p maps to the unique point on $H^+(E^+(S))$. T is well-defined. Also, by the Proposition 3.3.4 and the item 4 of the Proposition 3.3.5, T is onto. Moreover, following the proof of the claim in the part (II.) in the Theorem 5.2.2, T is 1-1 and continuous, too. Since S is a trapped set, $E^+(S)$ is non-empty and compact. Hence, $T(E^+(S)) = H^+(E^+(S))$ is non-empty and compact. However, it contradicts with the part (II.). The part (III.) is done.

We will show the part (IV.).

We first note that if F is a past-trapped set, then by a similar argument in the part (II.) and (III.), it suffices to show F is a past-trapped set. As $E^+(S)$ is closed and achronal, it is easy to show that $F = E^+(S) \cap \overline{J^-(\gamma)}$ is non-empty, closed and achronal. It leave us to show $E^+(F)$ is compact.

First, we will claim $E^-(F) \subseteq F \cup \partial J^-(\gamma)$. For any $x \in E^-(F) - F$, $E^-(F) \subseteq E^-(\overline{J^-(\gamma)})$ means x is either in $I^-(\gamma)$ or $\partial J^-(\gamma)$. For $x \in I^-(\gamma)$, there exists a past timelike curve $\alpha : [0, 1] \to M$ from some point $z = \alpha(0)$ on $J^-(\gamma)$ to $x = \alpha(1)$. On one hand, by the part (III.), we have $z \in \mathrm{int}(D^+(E^+(S)))$. On the other hand, it can be shown that $I^-(x) \cap E^+(S) = \emptyset$ and $x \notin E^+(S)$. Hence, $\alpha|_{(0,1)} \cap E^+(S)$ is non-empty and the intersection point is in F. As a result, we have $x \in I^-(F) \cap E^-(F)$. Contradiction. The claim is done.

Next, we let h and g be a complete Riemannian metric and the original Lorentzian metric on M respectively. Let $B = \{(x,v) \in TM | x \in F, g(v,v) = 0, h(v,v) = 1\}$. Since F is compact, a set $\{(x,v) \in TM | x \in F, h(v,v) = 1\}$ is compact. B is closed in the set. We have B is compact.

Then, we will show that there exists $K > 0$ such that for any affine parametrzied past null geodesic β with $(\beta(0), \beta'(0)) \in B$, we have $\beta((0,t]) \nsubseteq E^-(F) - F$ for $t \geq K$. Suppose the statement is false. There exists a sequence of affine parametrized past null geodesics β_i with $(\beta_i(0), \beta_i'(0)) \in B$ and $\beta(0,i] \subseteq E^-(F) - F$. We extend the β_i to be past endless. Since B is compact. W.L.O.G, we assume $(\beta_i(0), \beta_i'(0))$ converges to $(p,v) \in B$. By standard ODE theorems, there exists a past-endless null geodesic $\beta : [0,\infty) \to M$ with $\beta(0) = p$, $\beta'(0) = v$ and β_i converges to it. Then, by the above claim, we have $\beta(0,i] \subseteq \partial I^-(\gamma)$. For any $t \in [0,\infty)$, there exists $N > 0$ such that $\beta_i(t) \in \partial I^-(\gamma)$ for $i > N$. Since $\partial I^-(\gamma)$ is closed, we have $\beta[0,\infty) \subseteq \partial I^-(\gamma)$. Next, we extend $\beta : (-\infty, 0] \to M$ to be future endless null geodesic in a way such that $\beta(-\infty,\infty)$ is a single null geodesic. We will claim that $\beta(-\infty,\infty) \subseteq \partial I^-(\gamma)$. It suffices to show $\beta(-\infty, 0) \subseteq \partial I^-(\gamma)$. Suppose $\beta(t_0) \notin \partial I^-(\gamma)$ for some $t_0 < 0$. Since γ is a future endless timelike curve and $\beta(0) \in \partial(I^-(\gamma))$, we have $\beta(0) \notin \gamma$. By the Corollary 3.1.4, $\beta(t_0, 0) \cap \gamma$ is non-empty. Let say $\gamma(s_0)$ is an intersection point. $\beta(0) \preceq \gamma(s_0) \ll \gamma(s)$ for $s \geq s_0$. It means $\beta(0) \in I^-(\gamma)$ which contradicts with $\beta(0) \in \partial I^-(\gamma)$. The claim is done. However, by the condition (B), $\beta \subseteq \partial I^-(\gamma)$ has a pair of conjugate points. Let say they are $\beta(t_1)$ and $\beta(t_2)$ with $t_1 > t_2$. By the Theorem 4.2.12, it says there is a timelike relation between $\beta(t_1 + 1)$ and $\beta(t_2 - 1)$ which contradicts with the achronal property of $\partial I^-(\gamma)$. It means the sequence β_i don't exist.

Finally, we can show $E^-(F)$ is compact. With the same K as before, we define a map $T : B \times [0, K] \to M$ which maps $((p, v), t)$ to $\alpha(t)$ where α is an affine parametrized past null geodesic with $\alpha(0) = p$, $\alpha'(0) = v$ and it is . T is continuous and $B \times [0, K]$ is compact. Hence, $T(B \times [0, K])$ is compact. Also, it is easy to show that $E^-(F) = T(B \times [0, K]) \cap \partial J^-(F)$. It means $E^-(F)$ is compact. The part (IV.) is done.

We will prove the part (V.).

First, for each $n \in \mathbb{N}$, we let $a_n = \gamma(n)$. Then, we have $a_n \ll a_{n+1}$. Also, by the part (I.), M is strongly causal. $\{a_n\}$ has no converging subsequence, otherwise γ is not future endless. Similarly, we let $b_n = \lambda(-n)$. Then, $b_{n+1} \ll b_n$ and $\{b_n\}$ has no converging subsequence.

We will show that there exists a future timelike geodesic μ_n from b_n to a_n which meet $E^-(F)$ and μ_n is the longest among any past casual curve joining from b_n to a_n under the Lorentzian metric. It is clear that $b_n \in \text{int}(D^-(E^-(F)))$. Also, since $a_n \in \text{int}(D^+(E^+(S)))$, we have $a_n \in \text{int}(D^+(E^-(F)))$. Also, $b_1 \in D^-(E^-(F))$ means $b_1 \ll c$ for some $c \in F$. Since $I^+(b_1)$ is open, $b_1 \ll d$ for some $d \in J^-(\gamma)$. It means $b_1 \ll d \ll \gamma(k) = a_k$ for large k. W.L.O.G., we can assume $k = 1$. Hence, we have $b_n \ll b_1 \ll a_1 \ll a_n$. Also, by the Proposition 3.3.8, $\text{int}(D^-(E^-(F)))$ is globally hyperbolic. Then, by the Corollary 4.1.8, μ_n exists and its length is the longest. Also, since $a_n \in I^+(E^-(F)) \cap D^+(E^-(F))$ and $b_n \in I^-(E^-(F)) \cap D^-(E^-(F))$, we have $\mu_n \cap E^-(F)$ is non-empty, otherwise, it will contradict with the achronal property of $E^-(F)$.

Next, let the notations h and g be the same as in the part (IV.). μ_n is affine parametrized such that $\mu_n(0) \in E^-(F)$ and $h(\mu_n'(0), \mu_n'(0)) = 1$. Also, we can let $a_n = \mu_n(x_n)$ and $b_n = \mu_n(y_n)$ for some $y_n < 0 < x_n$. We will show $\mu_n|_{[y_n, x_n]}$ converges to an endless causal geodesic μ. We take $C = \{(p, v) \in TM | p \in E^-(F), g(v, v) \leq 0, h(v, v) = 1\}$. By a similar argument in the part (IV.), C is compact. Since $(\mu_n(0), \mu_n'(0)) \in C$, we can assume $(\mu_n(0), \mu_n'(0))$ converges to

$(p, v) \in C$. We let μ be an endless causal geodesic with $\mu(0) = p$ and $\mu'(0) = v$. Suppose x_n is bounded. There exists $R > 0$ such that $x_n \in [0, R]$ for all n. We can assume x_n converges to $x \in [0, R]$. Let d be a natural Riemaniann distance function between p, $q \in M$ with respect to h. For any $\epsilon > 0$, there exists N such that $d(\mu(x), \mu(x_n)) < \epsilon$ and $d(\mu(x_n), \mu_n(x_n)) < \epsilon$ for $n \geq N$. Then, for $n \geq N$, we have $d(\mu(x), a_n) \leq d(\mu(x), \mu(x_n)) + d(\mu(x_n), \mu_n(x_n)) < 2\epsilon$. Hence, a_n converges to $\gamma(x)$ which contradicts with the assumption of a_n. It means $x_n \to \infty$ as $n \to \infty$. Similarly, we have $y_n \to -\infty$ as $n \to \infty$.

Finally, we will show that μ_n is not the longest for some n which contradicts with the property of μ_n. By the conditions (B), μ has a pair of conjugate points. Let say they are $\mu(t_1 - 1)$ and $\mu(t_2 + 1)$ with $t_2 + 1 < t_1 - 1$. Then, by the Theorem 4.2.9 and 4.2.12, we have $\mu(t_2) \ll \mu(t_1)$. Let α be the future timelike curve joining from $\mu(t_2)$ to $\mu(t_1)$. There exists $\epsilon > 0$ such that

$$4\epsilon + L(\mu([t_2, t_1]) \leq L(\alpha) - - - (*).$$

There exists causally convex U and V such that $\mu(t_2) \in U$, $\mu(t_1) \in V$, $L(\alpha|_U) \leq \epsilon$ and $L(\alpha|_V) \leq \epsilon$. Let $\alpha(t'_2) \in \partial U \cap I^+(\mu(t_2))$ and $\alpha(t'_1) \in \partial V \cap I^-(\mu(t_1))$. Since $\mu_n(t_1)$ converges to $\mu(t_1)$ and $\mu_n(t_2)$ converges to $\mu(t_2)$ for n large, there exists $N > 0$ such that $\mu_n(t_2) \in I^-(\alpha(t'_2))$ and $\mu_n(t_1) \in I^+(\alpha(t'_1))$ for $n \geq N$. Hence, there exists a future timelike curves $\beta_n \subseteq U$ and $\eta_n \subseteq V$ joining from $\mu_n(t_2)$ to $\mu_n(t'_2)$ and $\mu_n(t'_1)$ to $\mu_n(t_1)$ respectively. Also, by the Theorem 4.1.7, for the same ϵ, we can make N larger such that for $n \geq N$

$$\begin{aligned} L(\mu_n[t_2, t_1]) &< L(\mu[t_2, t_1]) + \epsilon \\ &< L(\alpha[t_2, t_1]) - 3\epsilon \qquad \text{by (*).} \end{aligned}$$

Then, we have

$$\begin{aligned} L(\beta_n \cup \alpha[t'_2, t'_1] \cup \eta_n) &\geq L(\alpha[t'_2, t'_1]) \\ &\geq L(\alpha[t_2, t_1]) - 2\epsilon \\ &> L(\mu_n[t_2, t_1]) \qquad \text{for } n \geq N. \end{aligned}$$

In the third paragraph of the part (V.), we can make N larger such that such that for $n \geq N$, $a_n = \mu_n(x_n)$ with $x_n > t_1$ and $b_n = \mu(y_n)$ with $y_n < t_2$. Hence, we have

$$L(\mu_n[y_n, x_n]) < L(\mu_n[t_1, x_n]) + L(\beta_n \cup \alpha[t'_2, t'_1] \cup \eta_n) + L(\mu_n[t_2, y_n]).$$

It contradicts that μ_n is the longest. The part (V.) is done.

The lemma is done. □

We will show the singularity theorem in [6].

Theorem 5.3.3. *[6] Space-time (M, g) is timelike or null geodesic incomplete if*

(1.) there is no closed timelike curve [chronological condition];

(2.) $R_{ab}K^aK^b \geq 0$ for every causal vector K [strong energy condition];

(3.) any causal geodesic contains a point at which $\sum_{c,d=1}^{4} T^cT^dT_{[a}R_{b]cd[e}T_{f]} \neq 0$ where T is the tangent of the geodesic [generic condition];

(4.) there exists a compact achronal set without edge or a trapped surface.

Proof. Suppose the theorem is false. (M, g) is both timelike and null geodesic complete. By the Proposition 4.3.7, the conditions (2.) and (3.) in the theorem implies the conditions (B) in the Lemma 5.3.2. Also, we will claim that the condition (4.)implies the condition (C) is the Lemma 5.3.2.

In the case of the existence of a compact achronal set without edge which is called A, since edge$(A) = \emptyset$, by the Corollary 3.1.4, $\partial J^+(A) - A$ is generated by a past endless null geodesic lying in $\partial J^+(A)$. We have $E^+(A) = A$ which is compact. It means A is a future trapped set.

In the case of the existence of a trapped surface which is called Γ,in the proof of the part (I) of the Singularity Theorem 5.2.2, we have a map

$$\beta : \Gamma \times [0, b] \times \{1, 2\} \to M$$

such that $\beta(p, t, i)$ maps to $\gamma_i(t)$ with $\gamma_i(0) = p$. Then, $E^+(\Gamma) = \beta(\Gamma \times [0, b] \times \{1, 2\}) \cap \partial J^+(\Gamma)$ is a compact set.We will show $E^+(\Gamma) \cap \Gamma$ is a future trapped set. First, $E^+(\Gamma)$ is achronal and closed. $E^+(\Gamma) \cap \Gamma$ is achronal and closed. Also, it can be proved $E^+(\Gamma) \cap \Gamma$ is non-empty. Suppose it is empty. It means $\Gamma \subseteq I^+(\Gamma)$. Let $x \in \Gamma$, there exists a past timelike curve γ from x to $y \in \Gamma$. Then, we extend γ from y to z where $z \in I^-(y) \cap \Gamma$. As a result, we can extend γ in this way to become past-endless. However, M is strongly casual, there is no past endless casual curve which enters and re-enters infinitely many times in the compact set Γ. Contradiction. Finally, it suffices to show $E^+(E^+(\Gamma) \cap \Gamma)) = E^+(\Gamma)$. As Γ is a compact spacelike two-surface, we can cover it with a finite number of causally convex neighbourhood U_1, U_2, ...U_n where $U_i \cap \Gamma$ is achronal. It is easy to show that $I^+(E^+(\Gamma) \cap \Gamma) \subseteq I^+(\Gamma)$. For any $p \in I^+(\Gamma)$, then $p \in I^+(q_1)$ for some $q_1 \in \Gamma$. If $q_1 \in E^+(\Gamma) \cap \Gamma$, we have $p \in I^+(E^+(\Gamma) \cap \Gamma)$. If $q_1 \notin E^+(\Gamma) \cap \Gamma$, it means $q_1 \in I^+(q_2)$ for some $q_2 \in \Gamma$. W.L.O.G., we can assume $q_1 \in U_1$. Since $U_1 \cap \Gamma$ is achronal, W.L.O.G., we can assume $q_2 \in U_2 - U_1$. Then we repeat the above process. Since there is a finite number of U_i, we must have $p \in I^+(E^+(\Gamma) \cap \Gamma)$. Hence, we have $I^+(E^+(\Gamma) \cap \Gamma) = I^+(\Gamma)$. On the other hand, it is easy to show that $J^+(E^+(\Gamma) \cap \Gamma) \subseteq J^+(\Gamma)$. Then, for any $p \in J^+(\Gamma)$, if $p \in I^+(\Gamma)$, we have $p \in I^+(E^+(\Gamma) \cap \Gamma)$. If $p \notin I^+(\Gamma)$, $p \in J^+(\Gamma) - I^+(\Gamma)$, it means $p \in J^+(E^+(\Gamma) \cap \Gamma)$. Hence, we have $J^+(E^+(\Gamma) \cap \Gamma) = J^+(\Gamma)$. As a result, we have $E^+(\Gamma) = E^+(E^+(\Gamma) \cap \Gamma)$.

By the Lemma 5.3.2, (M, g) does not exist. The theorem is done. $\square$

Appendix

We will prove the limit curve theorem by using Arzela-Ascoli Theorem.

Theorem 5.3.4 (Arzela-Ascoli Theorem)**.** *Let(M, h) be a complete Riemannian manifold with distance function d_0 and $C([0,\infty))$ be a set of continuous function $f : [0, +\infty) \to M$. If a sequence $\{f_n\}$ in $C([0,\infty))$ satisfies that*

i. it is equicontinuous [i.e. for any compact $I \subseteq [0,\infty)$, for any $\epsilon > 0$, $\exists \delta > 0$, such that $d_0(f_n(x), f_n(y)) < \epsilon$ for $n \in \mathbb{N}$, $x, y \in I$ and $0 < |x - y| < \delta$];

ii. it is pointwise bounded [i.e. for $t \in [0,\infty)$, $sup\{d_0(f_n(t), f_1(t))|n \in \mathbb{N}\} < \infty$].

then there exists a $f \in C(\mathbb{R})$ and a subsequence of $\{f_n\}$ which converges to f uniformly on each compact subset $I \subseteq \mathbb{R}$.

Lemma 5.3.5. *M has a complete Riemannian metric h_0.*

Proof. M has an induced Riemannian metric h. Let $\{V_n\}$ be a sequence of compact sets in M such that $V_n \subseteq V_{n+1}$ and $M = \cup_{n=1}^{\infty} V_n$. For $n \geq 3$, we let $\chi_n : M \to \mathbb{R}$ be a smooth function such that $0 \leq \chi_n \leq 1$ on M and

$$\chi_n = \begin{cases} 1 & p \in \overline{V_n - V_{n-1}} \\ 0 & p \in M - V_{n+1} \text{ or } V_{n-2} \end{cases}$$

We let $d(x,y) : M \times M \to R$ by $d(x,y)$ =inf$\{\int \sqrt{h(\gamma', \gamma')}dt \; |\gamma$ is any piecewise differentiable curve from x to $y\}$. It is clear that d is continuous. There exists $\delta_n > 0$ such that $d(x,y) \geq \delta_n$ for $x \in \partial V_{n-1}$ and $y \in \partial V_n$. We let $h_0 = \sum \frac{1}{\delta_n} \chi_n h + g|_{V_2}$. We claim h_0 is complete. For any piecewise differentiable diverging curve $\alpha : [0,\infty) \to M$, we let $l(\alpha) = \int_0^{\infty} \sqrt{h_0(\alpha', \alpha')}dt$. For all n,

$l(\alpha) \geq l(\alpha|_{V_n}) = l(\alpha|_{V_2}) + l(\alpha|_{\overline{V_3 - V_2}}) + ... + l(\alpha|_{\overline{V_n - V_{n-1}}})$. For each terms, we have

$$\begin{aligned} l(\alpha|_{\overline{V_i - V_{i-1}}}) &\geq \int_{t_i}^{t_{i+1}} h_0(\alpha', \alpha')dt \\ &= \int_{t_i}^{t_{i+1}} \frac{1}{\delta_n} h(\alpha', \alpha')dt \\ &\geq 1. \end{aligned}$$

Thus, $l(\alpha)$ is unbounded. By Hopf-Rinow theorem, h_0 is complete. □

Lemma 5.3.6. *The length of any future inextendible piecewise differentiable curve γ with respect to the complete Riemannian metric h_0 is unbounded.*

Proof. We let $d_0 : M \times M \to \mathbb{R}$ as $d_0(x, y) = \inf\{\int \sqrt{h_0(\gamma', \gamma'} dt | \gamma$ is any piecewise differentiable curve from x to $y\}$. Suppose $l(\gamma)$ is bounded. We let $\gamma : [0, \infty) \to M$. Let $\{t_n\}$ be a sequence with $\lim_{n\to\infty} t_n = \infty$. For any $m > n$, $d_0(\gamma(t_n), \gamma(t_m)) \leq l(\gamma|_{[t_n, t_m]}) \to 0$ as $m, n \to \infty$. Hence, $\gamma(t_n)$ is a Cauchy Sequence. By Hopf-Rinow theorem, $\gamma(t_n)$ converges to some point $q \in M$. It is easy to show that $\lim_{t\to\infty} \gamma(t) = q$. Hence, γ is not future endless. Contradiction. □

Theorem 5.3.7. [*limit curve theorem*] *Let $\{\gamma_n\}$ be a sequence of future inextendible causal curves in (M, g). If p is an accumulation point of the sequence $\{\gamma_n\}$, then there is a future causal curve γ which is a limit curve of the sequence γ_n such that $p \in \gamma$ and γ is future inextendible.*

Proof. First, we assume γ_n is piecewise future endless differentiable causal curve. We show that there exists a limit curve of the sequence γ with $p \in \gamma$. By the Lemma 5.3.5 and 5.3.6, we can let γ_n is parametrized by the arc-length with respect to the complete Riemannian metric h_0 such that the domain of γ is $[0, \infty)$ with $\gamma_n(0) \to p$ as $n \to \infty$. Then, $d_0(\gamma_n(t_1), \gamma_n(t_2)) \leq l(\gamma_n|_{[t_1, t_2]}) = |t_2 - t_1|$. Hence, γ_n is equicontinuous on each compact set K on $\mathbb{R}$. Moreover, there exists $M > 0$ such that $d_0(\gamma_n(0), \gamma_1(0)) < M$ for all n. Hence, $d_0(\gamma_n(t), \gamma_1(t)) \leq$

$d_0(\gamma_n(t), \gamma_n(0)) + d_0(\gamma_n(0), \gamma_1(0)) + d_0(\gamma_1(t), \gamma_1(0)) \leq 2t + M$. Then, by the Theorem 5.3.4, there exists a subsequence γ_{n_k} of γ_n such that γ_{n_k} locally uniformly converges to γ with respect to d_0 and $\gamma_{n_k}(0) \to p$.

Then, we show that γ is future causal. W.L.O.G., we assume γ_n locally uniformly converges to γ. Let U be a convex normal neighbourhood of $\gamma(t)$. Then there exists $\epsilon > 0$, $N > 0$ such that $\gamma_n|_{[t-\epsilon, t+\epsilon]} \subseteq U$ for $n \geq N$. For any $t_1 < t_2 \in (t - \epsilon, t + \epsilon)$, for all $n \geq N$, there exists a future causal geodesic lying in U from $\gamma_n(t_1)$ to $\gamma_n(t_2)$. By smoothness of ODE, there is a future causal geodesic lying in U from $\gamma(t_1)$ to $\gamma(t_2)$. Hence, γ is a future causal curve.

Finally, we show γ is future endless. Suppose γ is not future endless. There is a future end-point $q = \lim_{t\to\infty} \gamma(t)$. Let g and T be the original Lorentzian metric and future direction in M. Let U be a convex normal neighbourhood of q such that

1. $\overline{U}$ is a compact set contanined in a single chart in (t, x_1, x_2, x_3);
2. $g = -dt^2 + dx_1 + dx_2 + dx_3$ at q;
3. If $\beta(s)$ is a future peicewise differentiable causal curve with respect to g and timelike vector field T, $\beta(s)$ is also a future timelike curve under $\widetilde{g} = -4dt^2 + dx_1^2 + dx_2^2 + dx_3^2$ and a timelike vector field $\frac{\partial}{\partial t}$.

We claim for any piecewise differentiable future causal curve $\alpha \subseteq U$, $l(\alpha)$ is globally bounded. We can reparametrize α as $\alpha(t) = (t, x_1(t), x_2(t), x_3(t))$ and $t \in [t_1, t_2]$.

$$
\begin{aligned}
l(\alpha) &= \int_{t_1}^{t_2} \sqrt{h_0(\alpha', \alpha')} dt \\
&\leq \int_{t_1}^{t_2} \sqrt{\lambda(1 + x_1'^2 + x_2'^2(t) + x_3'^2(t))} dt \qquad \lambda \text{ is maximum eignvalue of } h_0 \text{ in } \overline{U} \\
&\leq \int_{t_1}^{t_2} \sqrt{\lambda(1+4)} dt \\
&= \sqrt{5\lambda} |t_2 - t_1|.
\end{aligned}
$$

Since $\overline{U}$ is compact, the difference between $|t_2 - t_1|$ is bounded. The claim is done.

On the other hand, as q is future end-point of γ, there exists $t_0 > 0$ such that $\gamma|_{[t_0,\infty)} \subseteq U$. For any $\delta > 0$, there exists $N > 0$ such that $\gamma_n|_{[t_0,t_0+\delta]} \subseteq U$ for $n \geq N$. Since γ_n is arc-length parametrized with respect to h_0. Hence, for n tends to ∞, the length of the connected segment γ_n lying inside U tends to ∞. There is a contradiction with the claim.

As a result, the limit curve theorem is true for a sequence of piecewise differentiable future endless causal curve γ_n. In general, for any future endless causal curve η_n. We can assume the domain of η_n is $[0,\infty)$ with $\eta_n(0) \to p$. We cover $\eta_n|_{[0,1]}$ with $\{O_n(t) \subseteq B_{\frac{1}{n+1}}(\gamma(t))|t \in [0,1]$ and $O_n(t)$ is an open set of $\eta_n(t)\}$. We can cover it with $O_{n_1}(t), ..., O_{n_m}(t)$. Then, there is a future piecewise differentiable timelike curve $\gamma_n(t)$ from $\eta_n(0)$ to $\eta_n(1)$ which lying in $O_{n_1}(t) \cup ... \cup O_{n_m}(t)$. By induction, we can have a sequence of piecewise differentiable future endless causal curve γ_n. As γ_n converges to its limit curve γ, so does η_n. $\square$

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